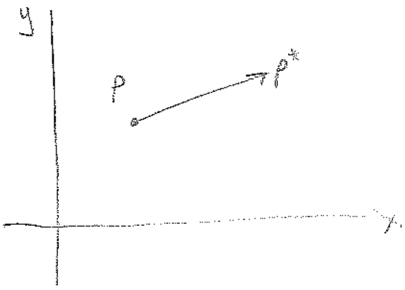
25-09-2008



(2)

$$T = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ p_0 & q_0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$= \begin{bmatrix} x + aw \\ y + bw \\ z + cw \\ 0 + sw \end{bmatrix}$$

$$T = \left[\begin{array}{c|c} \text{ROT} & \text{TRANS} \\ \hline \text{PERS} & \text{SCALE FACTOR} \end{array}\right]_{4\times4}$$

$s \geq 1$ shaing effect (the points will look nearer)

$0 \leq s < 1$ enlargement " (" " " look farther)

$$T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & l \\ \sin\theta & \cos\theta & 0 & m \\ 0 & 0 & 1 & n \\ & & & \end{bmatrix} \begin{matrix} P_x \\ P_y \\ P_z \\ \end{matrix}$$

coordinat frame representations (there are 6 of ...)

$$\begin{matrix} 1 \\ A = \end{matrix} \left[\begin{array}{ccc|c} 0 & 0 & -1 & l \\ 0 & 1 & 0 & m \\ -1 & 0 & 0 & n \\ 0 & 0 & 1 & 4 \end{array}\right]$$

$$T_6 = A_1 A_2 A_3 A_4 A_5 A_6$$

$$\underset{\substack{\text{w.r.t} \\ \text{world} \\ \text{coordinate} \\ \text{frame}}}{P} = T_6 \underset{\substack{\text{wrt } 6^{th} \\ \text{coordinate} \\ \text{frame}}}{(P')}$$

Certo

A_1 world coordinate
goto
A_6 ...
... ...

EE 587 (1) 23-09-2002

Name of the book: ROBOT MANIPULATORS
Mathematics, Programming and Control
Richard P. Paul
The MIT Press, 1981

(Not available, Only in the library)

Name of another Book: Robotics: Control, Sensing, Vision and Intelligence
K. S. FU, R. C. GONZALEZ, C.S.G. LEE
McGraw-Hill, 1987

VISION
OBSTACLE AVOIDANCE
COLLISION AVOIDANCE

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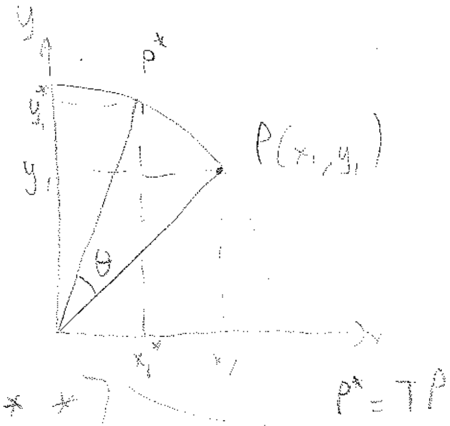
(1) 25-09-2002

1- Antropomorphical Robots (Human like)
2- Primal like Robots;
3- Industrial Robots;
4- Non-industrial Robots,

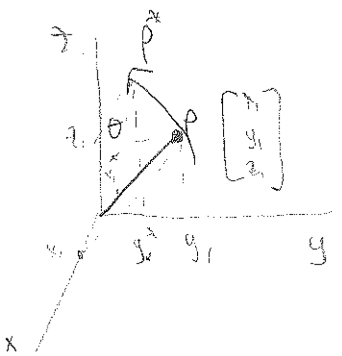
$$P = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad w \neq 1 \qquad \frac{x}{w} \quad \frac{y}{w} \quad \frac{z}{w}$$

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 4 \\ 4 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 24 \\ 24 \\ 24 \\ 12 \end{bmatrix}$$

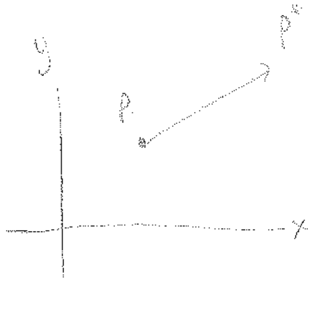


$P^* = TP$

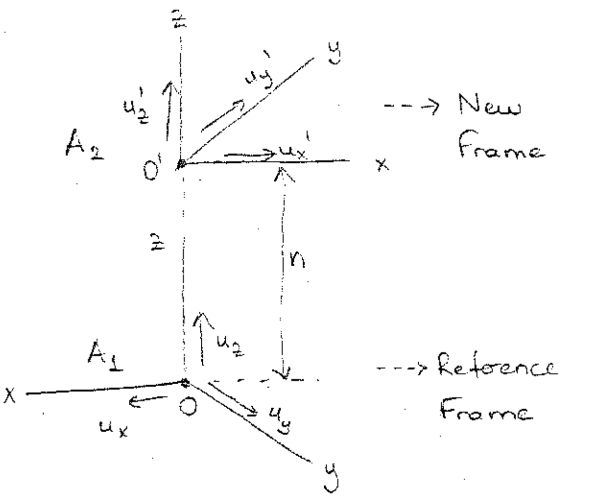
$$T = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$



$$T = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$



$P^* = TP$

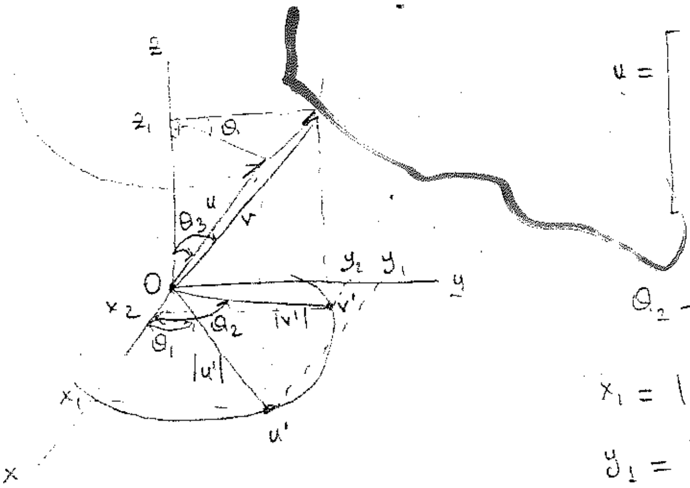


Transformation matrix:

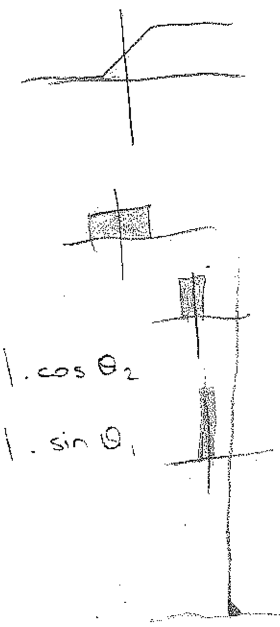
$$A_2 = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & n \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

· Translation: It's locating the new origin in terms of reference frame origin.

* Rotational Transformation:



$$u = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}$$



$$\theta_2 - \theta_1 = \theta$$

$$x_1 = |u'| . \cos\theta_1 \qquad x_2 = |v'| . \cos\theta_2$$

$$y_1 = |u'| . \sin\theta_1 \qquad y_2 = |v'| . \sin\theta_1$$

$$|u'| = |v'|$$

$$x_2 = |u'| \cos\theta_2 = |u'| . \cos(\theta_1 + \theta)$$

$$= |u'| . \cos\theta . \cos\theta_1 - |u'| . \sin\theta . \sin\theta_1$$

$$= x_1 \cos\theta - y_1 \sin\theta$$

$$y_2 = |u'| . \sin\theta_2 = |u'| \sin(\theta_1 + \theta)$$

$$= x_1 . \sin\theta + y_1 . \cos\theta$$

$$\Rightarrow \underbrace{\begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix}}_{v} = \underbrace{\left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{T} . \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}}_{u}$$

0 olmalı

(Rotation transformation about z-axis)

$$Rot(z, \theta) = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

---> Rotation transformation about z-axis with angle θ

0 olmalı

$$Rot(x,\phi)=\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \quad Rot(y,\alpha)=\left[\begin{array}{ccc|c} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Example. Given a point vector $u = 7i + 3j + 2k$, find the transformed point vector, v, rotated around z-axis by $90°$.

$$v = \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 2 \\ 1 \end{bmatrix}$$

If v is also rotated, about y-axis by $90°$ to give the transformed point vector w then

$$w = \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \cdot \begin{bmatrix} -3 \\ 7 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 3 \\ 1 \end{bmatrix}$$

$$\underline{w} = Rot(y, 90°) \cdot Rot(z, 90°) \cdot \underline{u}$$

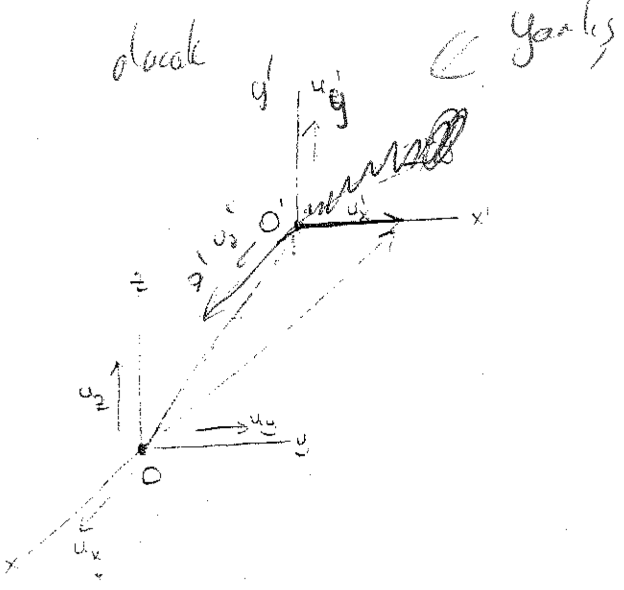
To obtain u from w:

$$\underline{u} = Rot^{-1}(z, 90°) \cdot Rot^{-1}(y, 90°) \cdot w$$
$$= Rot^{-1}(z, 90) \cdot Rot^{-1}(y, 90) \cdot \{Rot(y, 90) \cdot Rot(z, 90) \cdot u\}$$

First we apply inverse of the $Rot(y, 90)$ and then we apply inverse of $Rot(z, 90)$.

Coordinate Frames:

$$T = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

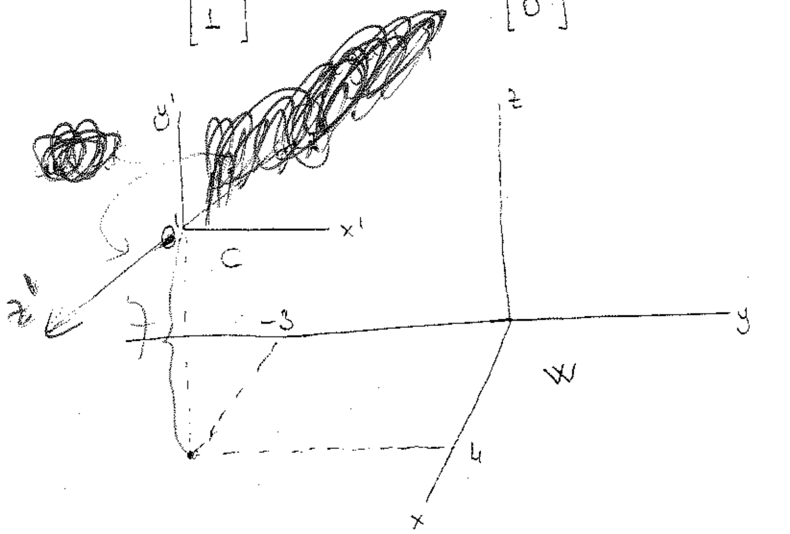


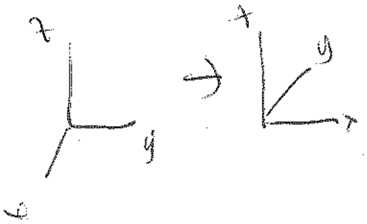
$$O' = T \cdot O$$

where $O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

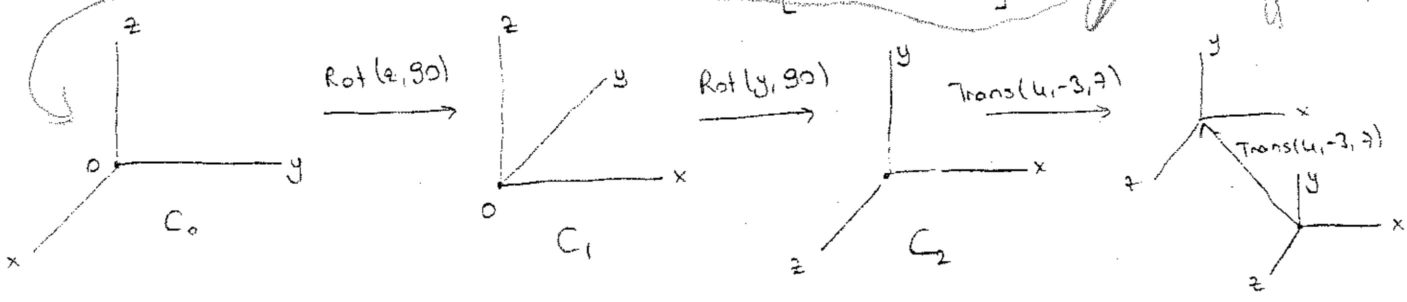
$$\Rightarrow O' = \begin{bmatrix} 4 \\ -3 \\ 7 \\ 1 \end{bmatrix} \quad \& \quad u'_x = \begin{bmatrix} 4 \\ -2 \\ 7 \\ 1 \end{bmatrix} \quad u'_x - O' = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u'_y = \begin{bmatrix} 4 \\ -3 \\ 8 \\ 1 \end{bmatrix} \quad u'_y - O' = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u'_z = \begin{bmatrix} 5 \\ -3 \\ 7 \\ 1 \end{bmatrix} \quad u'_z - O' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

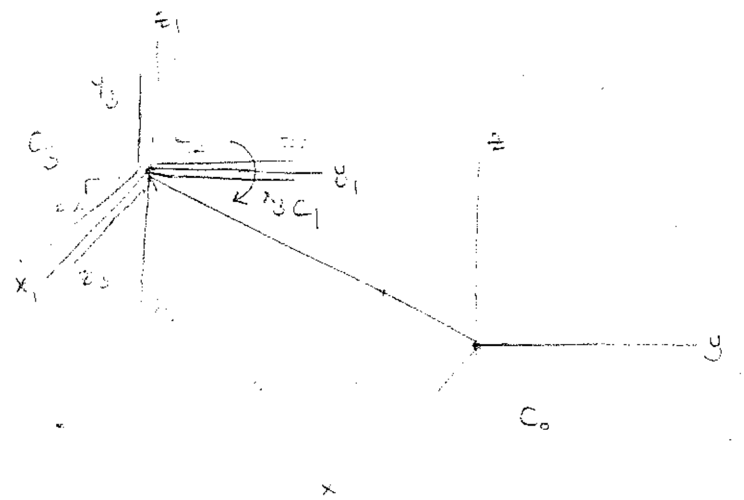




$$T = Trans(4,-3,7) \cdot Rot(y, 90°) \cdot Rot(z, 90°) = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$$



Now, let's look $\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} \cdot Transl(4,-3,7) \cdot Rot(y,90) \cdot Rot(z,90) = ?$



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So we obtain same result, i.e.

$$\begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} \cdot \underbrace{Transl(4,-3,7) \cdot Rot(y,90°) \cdot Rot(z,90)}_{(2^{nd} \text{ case})} = \underbrace{Transl(4,-3,7) \cdot Rot(y,90°) \cdot Rot(z,90)}_{(1^{st} \text{ case})} \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix}$$

Note that in 1^{st} case, we always do the rotation w.r.t. the base reference frame and in 2^{nd} case, we always do the rotation w.r.t. the last reference frame.

— o —

Given a coordinate frame C and transformation T:

(soldan) i. If frame C is postmultiplied by T to obtain $Y = C.T$, then the transformation is carried out w.r.t C.

(sağdan) ii. If frame C is premultiplied by T to obtain $X = T.C$, then the transformation is done w.r.t the base reference frame.

Example: Let $C = \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $T = Transl(10,0,0) \cdot Rot(z,90°)$

$$\Rightarrow T = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i. $Y = C.T$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 30 \\ 0 & 0 & -1 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii. $X_p = T.C$

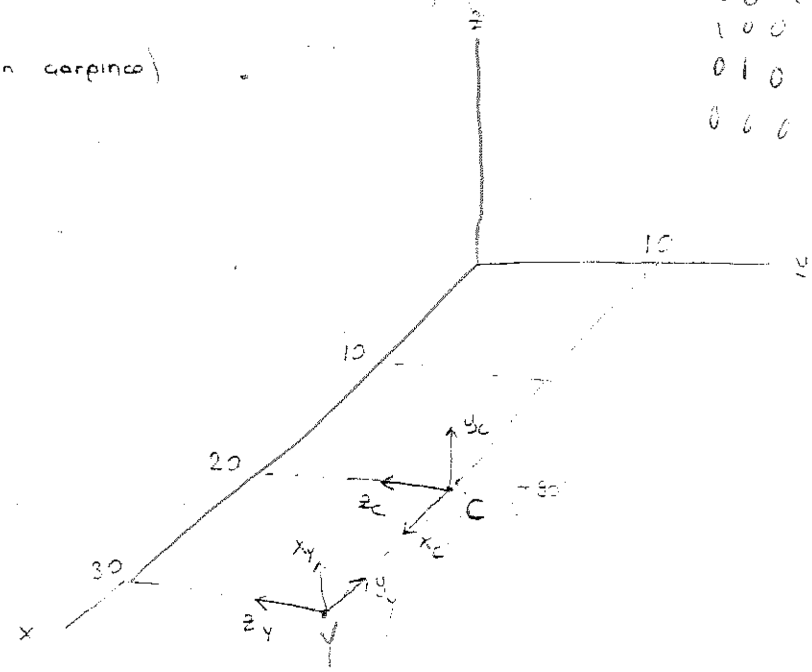
$$X_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 & 10 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

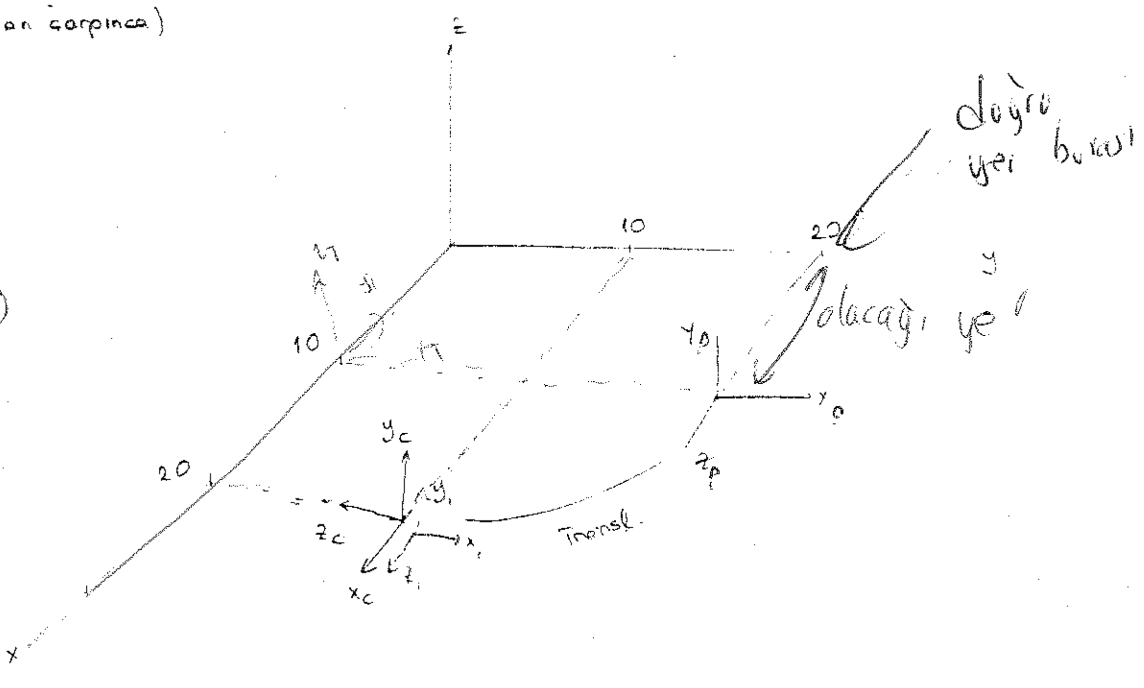
$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

niye böyle

i. (soldan çarpınca)

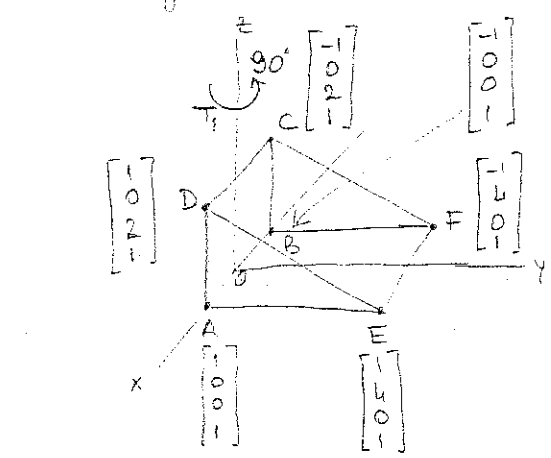


ii. (sağdan çarpınca)



02.10.2002 / Wednesday

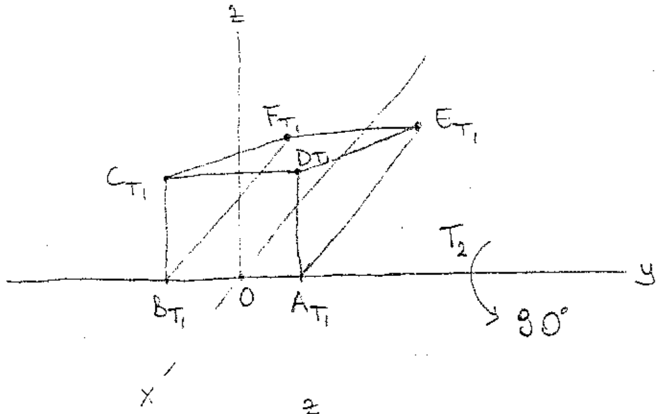
The Object:



$$OBJ = \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$T_1 = Rot(z, 90^\circ) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

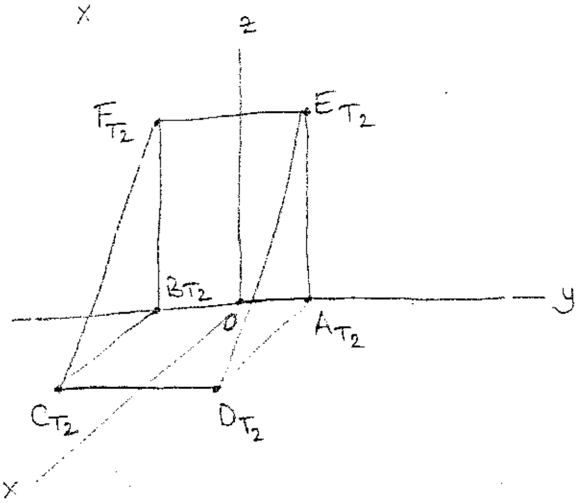
$$OBJ(T_1) = T_1 \cdot OBJ = \begin{array}{cccccc} A_{T_1} & B_{T_1} & C_{T_1} & D_{T_1} & E_{T_1} & F_{T_1} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & -4 & -4 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



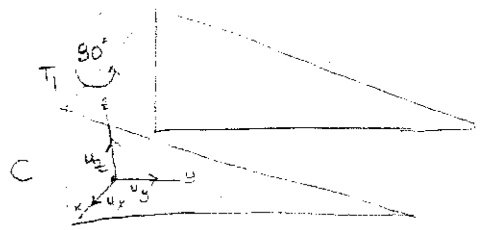
$$T_2 = Rot(y, 90^\circ)$$

$$(OBJ)_{T_2} = T_2 \, (OBJ)_{T_1}$$

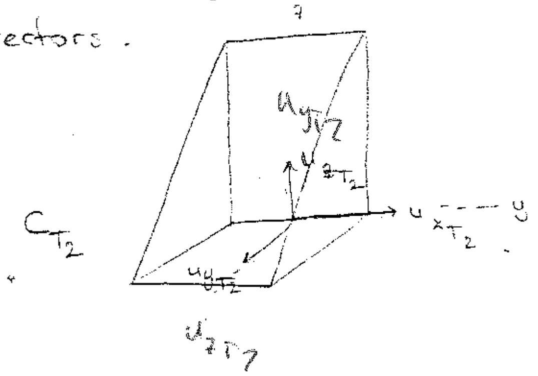
$$\Rightarrow (OBJ)_{T_2} = \begin{array}{cccccc} A_{T_2} & B_{T_2} & C_{T_2} & D_{T_2} & E_{T_2} & F_{T_2} \end{array} \\ \begin{bmatrix} 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

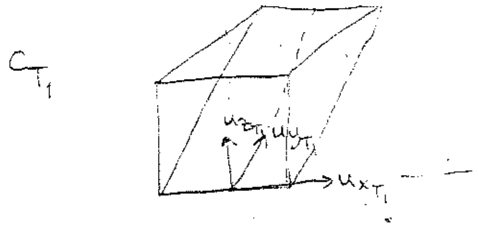


An another way to apply T_1 and T_2 to the given object is as follows:



We can define a coordinate frame in the object, and define the unit vectors of this frame. Then apply T_1 and T_2 to this unit vectors.





So just transforming the unit vector set gives the exact orientation of given object.

$$T = \begin{array}{c} \text{ROT} \qquad\qquad \text{TRANS} \\ \left[\begin{array}{ccc|c} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Note that;

i. The dot product of any row or column of $Rot(\cdot,\cdot)$ with any other row or column is zero, as the vectors are orthogonal.

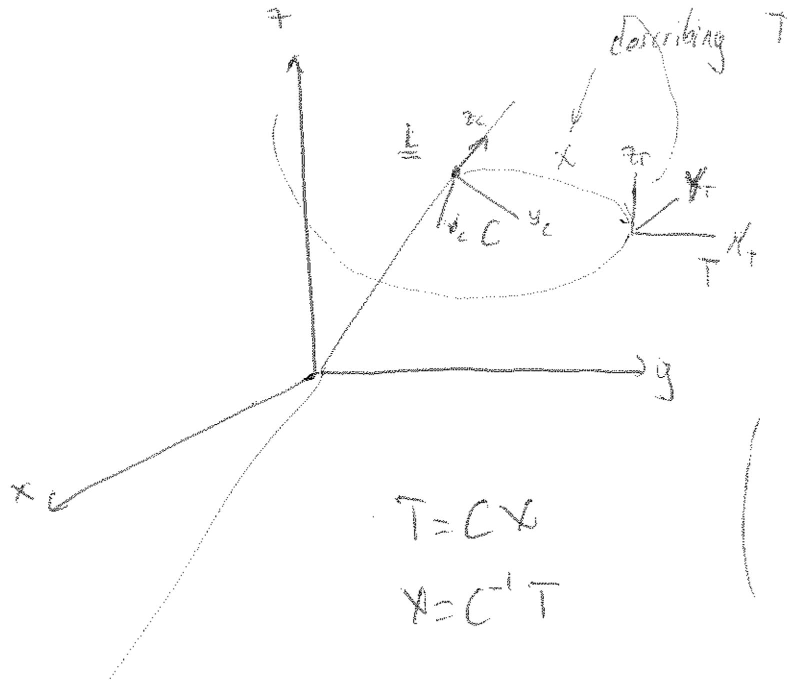
ii. The dot product of any row or column of $Rot(\cdot,\cdot)$ with itself is 1 as the vectors are of unit magnitude.

$$T = \underbrace{\begin{bmatrix} 1 & & & p_x \\ & 1 & & p_y \\ & & 1 & p_z \\ & & & 1 \end{bmatrix}}_{Transl(p_x, p_y, p_z)} \cdot \underbrace{\begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot(\cdot,\cdot)}$$

$$T^{-1} = Rot(\cdot,\cdot)^{-1} \cdot Transl(p_x, p_y, p_z)^{-1} = \begin{bmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & -p_x \\ & 1 & & -p_y \\ & & 1 & -p_z \\ & & & 1 \end{bmatrix}$$

$$\Rightarrow T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -p \cdot n \\ o_x & o_y & o_z & -p \cdot o \\ a_x & a_y & a_z & -p \cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

587 (1) 02-10-2002



$T = C X$

$X = C^{-1} T$

(iyi uğraş)

$$\underbrace{Rot(k,\theta)}\, T = C\, Rot(z_c,\theta)\, X$$

$$= \underbrace{C\, Rot(z_c,\theta)\, C^{-1}}\, T \qquad C = \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(k,\theta) = C\, Rot(z_c,\theta)\, C^{-1}$$

$$Rot(z_c,\theta)\, C^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & n_y & n_z & 0 \\ o_x & o_y & o_z & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (n_x\cos\theta - o_x\sin\theta) & (n_y\cos\theta - o_y\sin\theta) & (n_z\cos\theta - o_z\sin\theta) & 0 \\ (n_x\sin\theta + o_x\cos\theta) & (n_y\sin\theta + o_y\cos\theta) & (n_z\sin\theta + o_z\cos\theta) & 0 \\ a_x & a_y & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cancel{Rot(k,\theta) = \Big[\, n_x n_x \cos\theta - n_x o_x \sin\theta + n_x o_x \sin\theta + o_x o_x \cos}$$

$$Rot(k,\theta) = \begin{bmatrix} (n_x^2\cos\theta - n_x o_x \sin\theta + n_x o_x \sin\theta + o_x^2\cos\theta + d_x^2) & (n_x n_y \cos\theta - n_x o_y \sin\theta + n_y o_x \sin\theta + o_y o_x \cos\theta + d_x d_y) & (n_x n_z \cos\theta - n_x o_z \sin\theta + n_z o_x \sin\theta + o_z o_x \cos\theta + d_x d_z) & 0 \\ (n_y n_x \cos\theta - n_y o_x \sin\theta + n_x o_y \sin\theta + o_y o_x \cos\theta + d_y d_x) & (n_y^2\cos\theta - n_y o_y \sin\theta + n_y o_y \sin\theta + o_y^2\cos\theta + d_y d_y) & (n_y n_z \cos\theta - n_y o_z \sin\theta + n_z o_y \sin\theta + o_z o_y \cos\theta + d_y d_z) & 0 \\ (n_z n_x \cos\theta - n_z o_x \sin\theta + n_x o_z \sin\theta + o_z o_x \cos\theta + d_z d_x) & (n_z n_y \cos\theta - n_z o_y \sin\theta + n_y o_z \sin\theta + o_y o_z \cos\theta + d_z d_y) & (n_z n_z \cos\theta - n_z o_z \sin\theta + n_z o_z \sin\theta + o_z o_z \cos\theta + d_z d_z) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} n_x & n_y & n_z & 0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \\ 0 \end{bmatrix} = n_x^2 + n_y^2 + n_z^2 = 1 \qquad \begin{bmatrix} n_x & n_y & n_z & 0 \end{bmatrix} \begin{bmatrix} o_x \\ o_y \\ o_z \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} o_x & o_y & o_z & 0 \end{bmatrix} \begin{bmatrix} o_x \\ o_y \\ o_z \\ 0 \end{bmatrix} = 1 \qquad \begin{bmatrix} n_x & n_y & n_z & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} d_x & d_y & d_y & 0 \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ 0 \end{bmatrix} = 1$$

$$\begin{bmatrix} n_x & o_x & d_x \end{bmatrix} \begin{bmatrix} n_x \\ o_x \\ d_x \end{bmatrix} = 1 \qquad \begin{bmatrix} n_y & o_y & d_y \end{bmatrix} \begin{bmatrix} n_y \\ o_y \\ d_y \end{bmatrix} = 1 \qquad \begin{bmatrix} n_x & o_x & d_y & 0 \end{bmatrix} \begin{bmatrix} n_y \\ o_y \\ d_y \\ 0 \end{bmatrix} = 0$$

$$Rot(k,\theta) = \begin{bmatrix} (1-a_x^2)\cos\theta + a_x^2 & -a_x a_y \cos\theta - a_z \sin\theta + a_y a_x & -a_x a_z \cos\theta + a_y \sin\theta + a_z a_x & 0 \\ -a_x a_y \cos\theta + a_z \sin\theta + a_x a_y & (1-a_y^2)\cos\theta + a_y^2 & -a_y a_z \cos\theta - a_x \sin\theta + a_z a_y & 0 \\ -a_x a_z \cos\theta - a_y \sin\theta + a_x a_z & -a_y a_z \cos\theta + a_x \sin\theta + a_y a_z & (1-a_z^2)\cos\theta + a_z^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Define versine θ = vers θ = $1-\cos\theta$

let $k_x = a_x \quad k_y = a_y \quad k_z = a_z$

$$Rot(k,\theta) = \begin{bmatrix} k_x^2 \operatorname{vers}\theta + \cos\theta & k_y k_x \operatorname{vers}\theta - k_z \sin\theta & k_z k_x \operatorname{vers}\theta + k_y \sin\theta & 0 \\ k_x k_y \operatorname{vers}\theta + k_z \sin\theta & k_y k_y \operatorname{vers}\theta + \cos\theta & k_z k_y \operatorname{vers}\theta - k_x \sin\theta & 0 \\ k_x k_z \operatorname{vers}\theta - k_y \sin\theta & k_y k_z \operatorname{vers}\theta + k_x \sin\theta & k_z k_z \operatorname{vers}\theta + \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\theta = 90^\circ \quad \cos 90^\circ = 0 \quad \operatorname{vers} 90^\circ = 1$

$$Rot(k,90^\circ) = \begin{bmatrix} k_x k_x & k_y k_x - k_z & k_z k_x + k_y & 0 \\ k_x k_y + k_z & k_y k_y & k_z k_y - k_x & 0 \\ k_x k_z - k_y & k_y k_z + k_x & k_z k_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(k_x, 90^\circ)\Big|_{\substack{k_x=1\\k_y=0\\k_z=0}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(k,\theta) = R \Rightarrow \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} k_x^2 \,Vers\theta + C\theta & - & - & 0 \\ - & - & - & 0 \\ - & - & - & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$n_x + o_y + a_z + 1 = k_x^2\, Vers\theta + Cos\theta + k_y^2\, Vers\theta + Cos\theta + k_z^2\, Vers\theta + Cos\theta + 1$$

$$n_x + o_y + a_z = \underbrace{(k_x^2 + k_y^2 + k_z^2)}_{1} Vers\theta + 3Cos\theta = Vers\theta + 3Cos\theta$$

$$= (1 - Cos\theta) + 3Cos\theta = 1 + 2Cos\theta$$

$$\boxed{Cos\theta = \frac{1}{2}(n_x + o_y + a_z - 1)}$$

$\Rightarrow$ θ can be determined by this way

$$o_z - a_y = 2k_x Sin\theta \Rightarrow k_x = \frac{o_z - a_y}{2Sin\theta}$$

$$a_x - n_z = 2k_y Sin\theta \Rightarrow k_y = \frac{a_x - n_z}{2Sin\theta}$$

$$n_y - o_x = 2k_z Sin\theta \Rightarrow k_z = \frac{n_y - o_x}{2Sin\theta}$$

$$(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2 = 4Sin^2\theta$$

$$Sin\theta = \pm \frac{1}{2}\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}$$

$$Tan\theta = \frac{\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}}{(n_x + o_y + a_z - 1)}$$

* If θ is very small, then the axis of rotation is physically not well defined due to the small magnitude of both the numerator and denominator in k_x, k_y and k_z expressions. The vector k should be normalized to ensure that $|k| = 1$

* When the angle θ approaches 180° the vector k is once again poorly defined as the magnitude of $\sin\theta$ is again decreasing.
The axis of rotation is, however physically well defined in this case. When $\theta > 150^\circ$, denominators of k_x, k_y and k_z are less than 1.
As θ increases to 180° the rapidly decreasing magnitude of both numerators and denominators, lead to considerable inaccuracies in the determination of k. At $\theta = 180^\circ$ eqns are of the form $\frac{0}{0}$, yielding no information at all about a physically well defined vector k.

If $\theta > 90^\circ$ the we must follow a different approach in determining k.

$$k_x^2 \operatorname{Vers}\theta + \cos\theta = n_x$$

$$k_y^2 \operatorname{Vers}\theta + \cos\theta = o_y$$

$$k_z^2 \operatorname{Vers}\theta + \cos\theta = a_z$$

Substituting

$$\cos\theta = \frac{1}{2}(n_x + o_y + a_z - 1)$$

into the above set of equations, and solving for the elements of k we obtain

$$k_x = \pm\sqrt{\frac{(n_x - \cos\theta)}{(1-\cos\theta)}} \qquad k_y = \pm\sqrt{\frac{(o_y - \cos\theta)}{(1-\cos\theta)}} \qquad k_z = \pm\sqrt{\frac{(a_z - \cos\theta)}{(1-\cos\theta)}}$$

The largest component of k defined by these eqns corresponds to the most positive component of n_x, o_y, and a_z

The sine of θ (note that $\theta > 90^\circ$) must be true, then the sign of the component of k defined above is determined by the sign of corresponding term within the parenthesis as $(o_z - a_y)$, $(a_x - n_z)$ and $(n_y - o_x)$. Since

$$\left.\begin{aligned} (o_z - a_y) &= 2k_x \sin\theta \\ (a_x - n_z) &= 2k_y \sin\theta \end{aligned} \quad \wedge \quad (n_y - o_x) = 2k_z \sin\theta \right\} \text{ where } \sin\theta > 0$$

Therefore

$$k_x = \operatorname{Sgn}(o_z - a_y)\sqrt{\frac{n_x - \cos\theta}{1-\cos\theta}} \qquad k_y = \operatorname{Sgn}(a_x - n_z)\sqrt{\frac{o_y - \cos\theta}{1-\cos\theta}}$$

$$k_z = \operatorname{Sgn}(n_y - o_x)\sqrt{\frac{a_z - \cos\theta}{1-\cos\theta}}$$

where $\operatorname{sgn}(e) = +1$ if $e \geq 0$
$= -1$ if $e < 0$

Only the largest element of k can be determined from above eqns, The remaining elements are more accurately determined by the eqns formed as summed pairs of of-diagonal elements of k and $Rot(k,\theta)$ matrices. Thus

$$n_y + o_x = 2 k_x k_y \operatorname{Vers}\theta$$

$$o_z + a_y = 2 k_y k_z \operatorname{vers}\theta$$

$$n_z + a_x = 2 k_z k_x \operatorname{Vers}\theta$$

If k_x results to be the larges

$$k_y = \frac{n_y + o_x}{2k_x \operatorname{Vers}\theta} \qquad k_z = \frac{n_z + a_x}{2k_x \operatorname{Vers}\theta}$$

If k_y results to be the largest

$$k_x = \frac{n_y + o_x}{2k_y \operatorname{Vers}\theta} \qquad k_z = \frac{o_z + a_y}{2k_y \operatorname{Vers}\theta}$$

If k_z results to be the largest then

$$k_x = \frac{n_z + a_x}{2k_z \operatorname{Vers}\theta} \qquad k_y = \frac{o_z + a_y}{2k_z \operatorname{Vers}\theta}$$

The algorithm to determine θ and k is then as follows

(1) find θ

$$\cos\theta = \frac{1}{2}(n_x + o_y + a_z - 1) \quad , \quad \sin\theta = \pm\frac{1}{2}\sqrt{(o_z - a_y)^2 + (a_x - n_z)^2 + (n_y - o_x)^2}$$

(2) If $\theta < 90°$ then find

$$k_x = \frac{o_z - a_y}{2\sin\theta} \quad k_y = \frac{a_x - n_z}{2\sin\theta} \quad k_z = \frac{n_y - o_x}{2\sin\theta}$$

(3) If $\theta > 90°$ then check $(o_z - a_y)$, $(a_x - n_z)$ and $(n_y - o_x)$

(a) If $(o_z - a_y)$ is the largest then

$$k_x = \mathrm{Sgn}(o_z - a_y)\sqrt{\frac{n_x - \cos\theta}{1 - \cos\theta}}, \quad k_y = \frac{n_y + o_x}{2k_x \operatorname{vers}\theta} \quad k_z = \frac{n_z + a_x}{2k_x \operatorname{vers}\theta}$$

(b) If $(a_x - n_z)$ is the largest then

$$k_y = \mathrm{Sgn}(a_x - n_z)\sqrt{\frac{o_y - \cos\theta}{1 - \cos\theta}}, \quad k_x = \frac{n_y + o_x}{2k_y \operatorname{vers}\theta} \quad k_z = \frac{o_z + a_y}{2k_y \operatorname{vers}\theta}$$

(c) If $(n_y - o_x)$ is the largest then

$$k_z = \mathrm{sgn}(n_y - o_x)\sqrt{\frac{a_x - \cos\theta}{1 - \cos\theta}}, \quad k_x = \frac{n_z + a_x}{2k_z \operatorname{vers}\theta}, \quad k_y = \frac{o_z + a_y}{2k_z \operatorname{vers}\theta}$$

Fossel sor

Example: Determine the ~~equivalent axis and the rotation or equivalent~~ axis of rotation and the equivalent rotation angle for the matrix given

$$Rot(y, 90°)\,Rot(z, 90°) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

S87 (8) 07-10-2002

First determine θ as

$\cos\theta = \frac{1}{2}(0+0+0-1) = -\frac{1}{2}$ θ may be 120° or 240°

then

$\sin\theta = \frac{1}{2}\sqrt{(1-0)^2+(1-0)^2+(1-0)^2} = \frac{\sqrt{3}}{2}$ θ may be 60° or 120°

hence

$\theta = 120^\circ$

As $\theta > 90^\circ$, determine the largest component of $\bar{k}$ corresponding to the largest element on the diagonal. As all diagonal elements are equal in this example we may pick any one of them. Pick k_x first

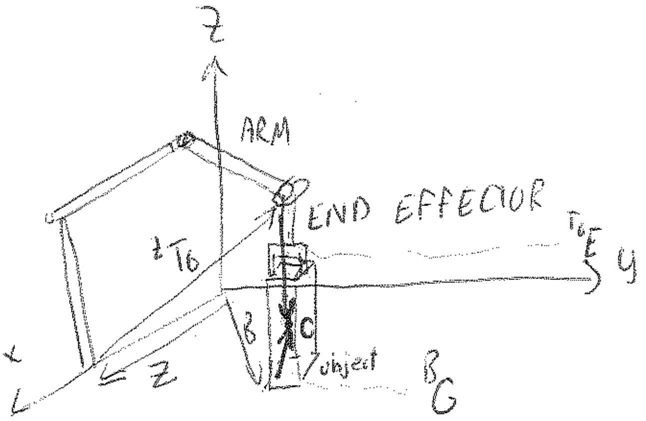
$$k_x = \mathrm{sgn}(1-0)\sqrt{\frac{0-\cos 120^\circ}{1-\cos 120^\circ}} = +\frac{1}{\sqrt{3}}$$

$$k_y = \frac{1+0}{2\frac{1}{\sqrt{3}}(1-\cos 120^\circ)} = \frac{1}{\sqrt{3}} \qquad k_z = \frac{0+1}{2\frac{1}{\sqrt{3}}(1-\cos 120^\circ)} = \frac{1}{\sqrt{3}}$$

$$k = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k \qquad \mathrm{Rot}(k, 120^\circ) = \mathrm{Rot}(y, 90^\circ)\,\mathrm{Rot}(z, 90^\circ)$$

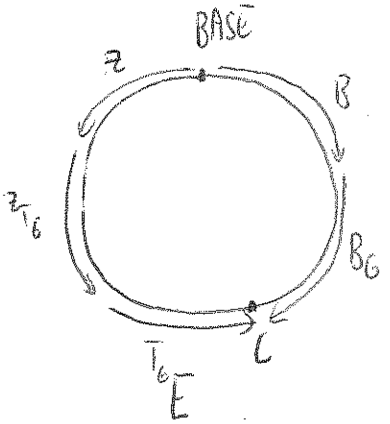
587 ⓫ 09-10-2002

Transform Equations



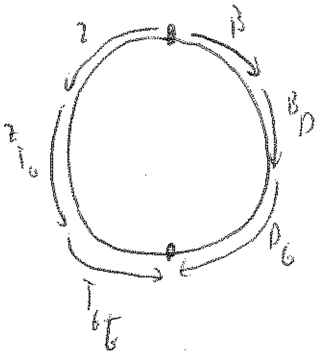
$$\underbrace{{}^{0}Z\ {}^{Z}T_6\ {}^{T_6}E\ [\ ]}_{C} = \underbrace{{}^{0}B\ {}^{B}G\ [\ ]}_{C}$$

^{E}C ^{G}C

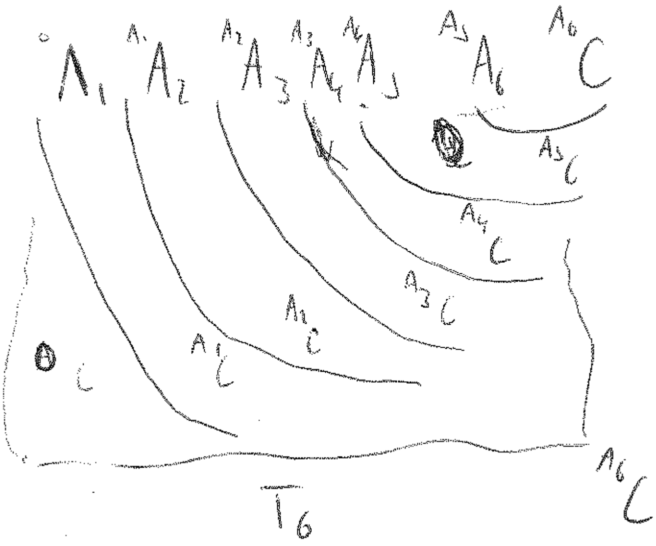


$$Z\ {}^{Z}T_6\ {}^{T_6}E = B\ {}^{B}G$$

$$Z\ {}^{Z}T_6\ {}^{T_6}E\ ({}^{B}G)^{-1} = B$$



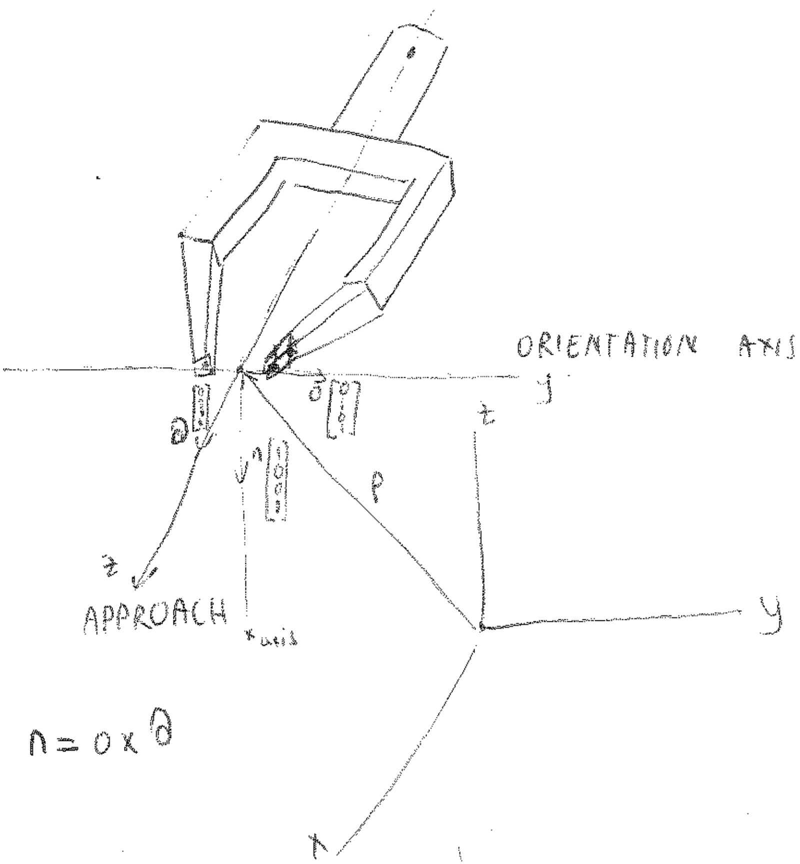
$$\boxed{Z\ {}^{Z}T_6\ {}^{T_6}E = B\ {}^{B}D\ {}^{D}G}$$



$$\boxed{C = T_6\ {}^{A_6}C}$$ TOTAL ARM REPRESENTATION

Provided that arm is made of 6 degree of freedome

$$\begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



WORLD COORD FRAME

$n = o \times a$

$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\uparrow n \quad \uparrow o \quad \uparrow a \quad \uparrow p$

$n = o \times a$

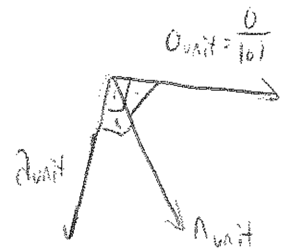
we have to find 6 components for orientation $(o_x, o_y, o_z, a_x, a_y, a_z)$

* and 3 components for position

* (p_x, p_y, p_z). we can find n by cross product of o and a

$n = o \times a$

$\frac{a}{|a|} = a_{unit}$



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$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = A_1 A_2 A_3 A_4 A_5 A_6$$

Eqns Relating End-Effector Position and Orientation in base coordinates to Joint variables in Stanford / JPL Arm.

$$p_x = d_6 [\underbrace{s_5}_{\sin(\theta_5)} (s_1 s_4 + \underbrace{c_1}_{\cos(\theta_1)} c_2 s_4) + c_1 s_2 c_5] + d_3 c_1 s_2 - d_2 s_1$$

$$p_y = d_6 [s_5 (-c_1 c_4 + s_1 c_2 s_4) + s_1 s_2 c_5] + d_3 s_1 s_2 + d_2 c_1$$

$$p_z = d_6 (c_2 c_5 - s_2 s_4 s_5) + d_3 c_2 + d_1$$

$$a_x = s_5 (s_1 c_4 + c_1 c_2 s_4) + c_1 s_2 c_5$$

$$a_y = s_5 (-c_1 c_4 + s_1 c_2 s_4) + s_1 s_2 c_5$$

$$a_z = c_2 c_5 - s_2 s_4 s_5$$

$$o_x = -s_6 [c_5 (s_1 c_4 + c_1 c_2 s_4) - c_1 s_2 s_5] + c_6 [-s_1 s_4 + c_1 c_2 c_4]$$

$$o_y = -s_6 [c_5 (-c_1 c_4 + s_1 c_2 s_4) - s_1 s_2 s_5] + c_6 [c_1 s_4 + s_1 c_2 c_4]$$

$$o_z = c_2 s_5 s_6 + s_2 s_4 c_5 s_6 - c_4 s_2 c_6$$

$$n_x = c_6 [c_5 (s_1 c_4 + c_1 c_2 s_4) - c_1 s_2 s_5] + s_6 [-s_1 s_4 + c_1 c_2 c_4]$$

$$n_y = c_6 [c_5 (-c_1 c_4 + s_1 c_2 s_4) - s_1 s_2 s_5] + s_6 [c_1 s_4 + s_1 c_2 c_4]$$

$$n_z = -c_2 s_5 c_6 - s_2 [s_4 c_5 c_6 + c_4 s_6]$$

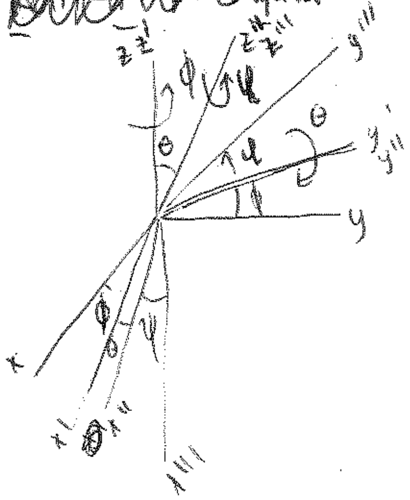
Example:
The configuration of a Stanford/JPL manipulator is such that the joint variables have the following values w.r.t to the reference positions

$\theta_1 = \frac{\pi}{4}$ rad $\quad \theta_2 = 0 \quad d_3 = 20$ cm $\quad \theta_4 = \frac{\pi}{2}$ rad $\quad \theta_5 = \frac{\pi}{3}$ rad $\quad \theta_6 = 0$

$$T_6 = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{d_6\sqrt{3}}{2\sqrt{2}} - \frac{d_2}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{d_6\sqrt{3}}{2\sqrt{2}} + \frac{d_2}{\sqrt{2}} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & \frac{d_6}{2} + d_3 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

EULER ANGLE $(\phi, \theta, \psi) = Rot(z,\phi)\, Rot(y,\theta)\, Rot(z,\psi)$

$$Rot(y,\theta) \downarrow Rot(y'',\theta) \qquad Rot(z,\psi) \downarrow Rot(z''',\psi)$$



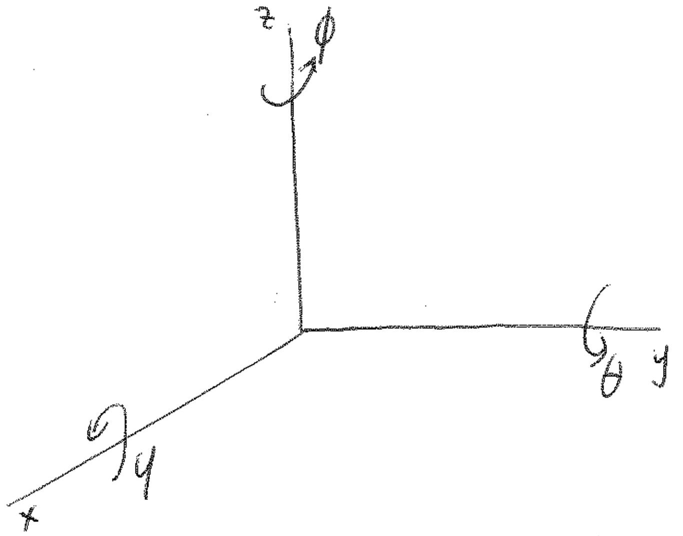
$$= Rot(z,\phi) \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi) & (-\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi) & \cos\phi\sin\theta & 0 \\ (\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi) & (-\sin\phi\cos\theta\sin\psi - \cos\phi\cos\psi) & \sin\phi\sin\theta & 0 \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Roll – Pitch – Yaw

$$RPY_i(\phi, \theta, \psi) = Rot(z, \phi)\, Rot(y, \theta)\, Rot(x, \psi)$$

x = TC



1. Midterm (4 Kasım)
2. Midterm (16 Aralık)

$$RPY(\phi, \theta, \psi) = \begin{bmatrix} \cos\phi\cos\theta & (\cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi) & (\cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi) & 0 \\ \sin\phi\cos\theta & (\sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi) & (\sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi) & 0 \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

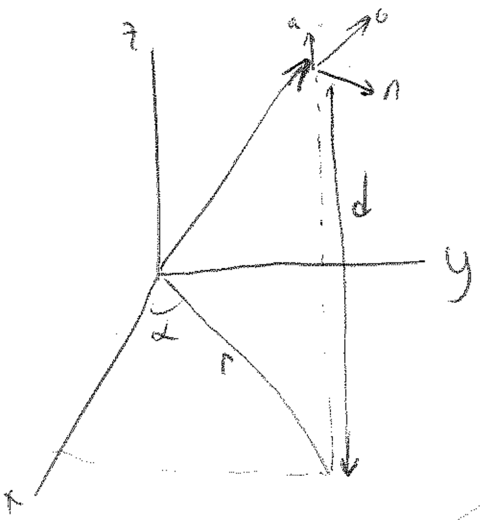
Example:

The orientation of the end-effector is given in a cartesian coordinate system by angles $\phi = 0$, $\theta = \frac{\pi}{6}$ rad $\psi = \frac{\pi}{2}$ rad representing the roll pitch and yaw respectively. Determine the orientation of the gripper in the cartesian coordinates by specifying vectors n, o, and a

$$\begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = RPY\left(0, \frac{\pi}{6}, \frac{\pi}{2}\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

CYLINDRICAL POSITION SPECIFICATION

This is the way of specifying the position of a robot-arm in cylindrical coordinates.



$$cyl(d, \alpha, r) = Trans(0,0,d)\, Rot(z,\alpha)\, Trans(r,0,0)$$

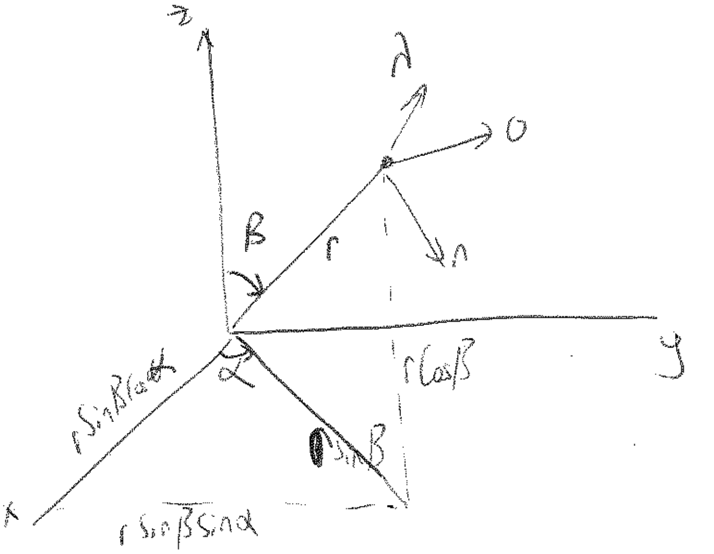
$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & r\cos\alpha \\ \sin\alpha & \cos\alpha & 0 & r\sin\alpha \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* If we rotate this matrix $(-\alpha)$ degrees what happens

$$\underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & r\cos\alpha \\ \sin\alpha & \cos\alpha & 0 & r\sin\alpha \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) & 0 & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T} = \begin{bmatrix} 1 & 0 & 0 & r\cos\alpha \\ 0 & 1 & 0 & r\sin\alpha \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

? ? Niye sağına matrise yazdı

SPHERICAL COORDINATES



$$Sph(\alpha, \beta, r) = Rot(z, \alpha)\, Rot(y, \beta)\, Trans(0, 0, r)$$

$$= \begin{bmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta & r\cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta & r\sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

* Turn y $(-\beta)$ and $z(-\alpha)$ angles

$$Sph(\alpha, \beta, r)\, Rot(y, -\beta)\, Rot(z, -\alpha) = \begin{bmatrix} 1 & 0 & 0 & r\cos\alpha\sin\beta \\ 0 & 1 & 0 & r\sin\alpha\sin\beta \\ 0 & 0 & 1 & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SPECIFICATION OF A MATRICES

For an n degree of freedom manipulator, there will be n links and n joints. The base of the manipulator is link 0 and is not considered one of the six links, Link 1 is connected to the base link over joint 1 and there is no joint at the end of the final link.

Any link can be characterized by two dimensions.

The common normal length a_n;

The angle α_i (between two successive joint axis in a plane $\perp$ to a_i)

It is customary to call a_i "the length" and α_i "the twist" of the link.

Each link will have one normal at each joint. At each joint axis there are two normals (namely x_{i-1} and x_i) pertaining the two links connected to it.

The relative position of two such connected links is given by d_i, the distance between the normals along the joint axis i, and θ_i, the angle between the normals measured in a plane normal to the axis.

i.e The angle between x_i and x_{i-1} d_i and θ_i are called "the distance" and "the angle" between the links, respectively.

In assigning the coordinate frame at joint (i) the origin of the frame of link i is set to be at the intersection of the common normal between the axes of joints (i) and $(i+1)$

In the case of intersecting joint axes, the origin is at the point of intersection of the joint axes.

If the axes are parallel, the origin is chosen to make the joint distance zero for the next link whose coordinate origin is defined.

The z-axis for link i will be aligned with the axis of joint i+1

The x axis will be aligned with any common normal which exists and is directed along the normal from joint i to joint (i+1). In the case of intersecting joints the direction of the x-axis is // (parallel) or antiparallel to the vector cross product ($z_{i-1} * z_i$). Notice that this condition is also satisfied for the x-axis directed along the normal between joints i and (i+1)

θ_i is zero for the i'th revolute joint when x_{i-1} and x_i are parallel and have the same direction

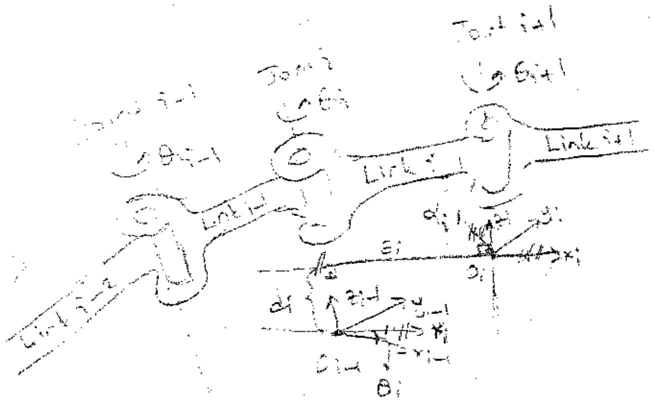
$$Sph(\alpha, \beta, r) = Rot(z, \alpha) \cdot Rot(y, \beta) \cdot Trans(0, 0, r)$$

$$= \begin{bmatrix} \cos\alpha \cos\beta & -\sin\alpha & \cos\alpha \sin\beta & r\cos\alpha \sin\beta \\ \sin\alpha \cos\beta & \cos\alpha & \sin\alpha \sin\beta & r\sin\alpha \sin\beta \\ -\sin\beta & 0 & \cos\beta & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Sph(\alpha, \beta, r)\, Rot(y, -\beta) \cdot Rot(z, -\alpha) = \begin{bmatrix} 1 & 0 & 0 & r\cos\alpha \sin\beta \\ 0 & 1 & 0 & r\sin\alpha \sin\beta \\ 0 & 0 & 1 & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

16.10.2002 / Wednesday

DETERMINATION OF A MATRICES:



For an n d.o.f manipulator, there will be n link and n joints. The base of the manipulator is link 0 and is not considered one of the six (n) links. Link 1 is connected to the base link over joint 1. There is no joint at the end of the final link.

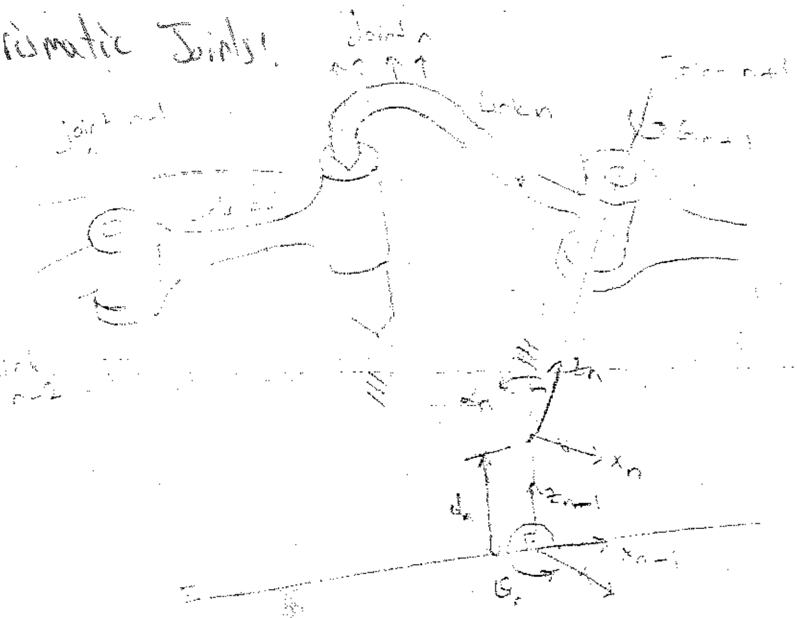
Any link can be characterized by two dimensions;

- the common normal length, a_i ;
- the angle α_i (between the two successive joint axes in a plane ⊥. to a_i)

It's customary to call a_i "the length", and α_i "the twist" of the link.

Each link will have one normal at each joint. At each joint axis, there are two normals (namely x_{i-1} and x_i) pertaining the two links connected to it. The relative position of two such connected links is given by d_i, the distance between the normals along the joint axis i, and θ_i the angle between the normals measured in a plane normal to the axis, i.e., the angle between x_i and x_{i-1}.

Prismatic Joints!



Distance d_n is the joint variable. The direction of the joint axis is the direction in which the joint moves.

The length a_n has no meaning and it is set to zero.

The origin of the coordinate frame for a prismatic joint is coincident with the next defined link origin. The z-axis of the prismatic link is aligned with the axis of joint (n+1). The x_n axis is // or antiparallel to the vector cross product of the direction of the prismatic joint and z_n.

For a prismatic joint we will define the zero position when $d_n = 0$.

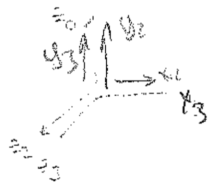
In defining a coordinate frame for a link n with a prismatic joint n:

- set $a_n = 0$
- origin: it is same as the next defined origin of the next link.
- x-axis: it is // to $\pm(z_n \times z_{n-1})$ where z_{n-1} is the axis of prismatic joint.
- z-axis: it is aligned with joint axis (n+1)
- joint variable: it is d_n.

$$A_n = Rot(z_{n-1}, \theta_n) \cdot Transl(0, 0, d_n) \cdot Rot(x_n, \alpha_n)$$

$$A_n = \begin{bmatrix} \cos\theta_n & -\sin\theta_n \cdot \cos\alpha_n & \sin\theta_n \cdot \sin\alpha_n & 0 \\ \sin\theta_n & \cos\theta_n \cdot \cos\alpha_n & -\cos\theta_n \cdot \sin\alpha_n & 0 \\ 0 & \sin\alpha_n & \cos\alpha_n & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Prismatic Joint Representation}$$

Link 3



Joint axes 3 and 4 parallel. z_3 is joint axis 4. x_3 is aligned with the common normal (between z_2 and z_3)

$\theta = \theta_3$, $a = a_3$, $d_3 = 0$, $\alpha_3 = 0$

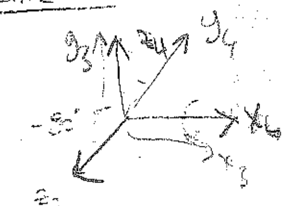
$$A_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Rotation about z_3 axis which is old z_2

→ Translation along both x_3 and y_3

$$\begin{pmatrix} c_3 & -s_3 & 0 & 0 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Link 4



Origin is at the intersection point of common normal between joints 5 and 4. The joint axis is z_4. x_4 is aligned with $\pm(z_3 \times z_4)$.

$\theta = \theta_4$, $a = a_4$, $d_4 = 0$, $\alpha_4 = -90°$

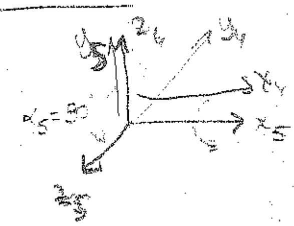
$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & a_4 c_4 \\ s_4 & 0 & c_4 & a_4 s_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about y_4 axis which is old z_3

→ Translation along both x_4 and y_4

$$\begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_4 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{matrix} c_4 & 0 & -s_4 & a_4 c_4 \\ s_4 & 0 & c_4 & a_4 s_4 \\ 0 & -1 & 0 & 0 \end{matrix}$$

Link 5:



Joint axes 5 and 6 are intersecting. z_5 is the joint axis 6.

x_5 is // to $\pm(z_4 \times z_5)$

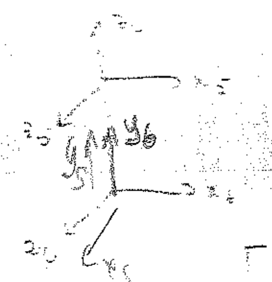
$\theta = \theta_5$, $a_5 = 0$, $d_5 = 0$, $\alpha_5 = 90°$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Rotation about y_5 which is old z_4.

$$\begin{pmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \end{bmatrix}$$

Link 6:



~~Origin is same as for link 5.~~ z_6 is the joint axis.

~~x_6 is // to $\pm(z_4 \times z_5)$~~ x_6 aligned with the common normal (btw z_5 and z_6)

$\theta = \theta_6$, $a_6 = 0$, $d_6 = 0$, $\alpha_6 = 0$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} c_2 & -s_2 & 0 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} c_1 & s_1 & 0 & -l_1 \\ 0 & 0 & 1 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A_1^{-1}} \cdot \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (n_x c_1 + n_y s_1) & (o_x c_1 + o_y s_1) & (a_x c_1 + a_y s_1) & (p_x c_1 + p_y s_1 - l_1) \\ n_z & o_z & a_z & p_z \\ (n_x s_1 - n_y c_1) & (o_x s_1 - o_y c_1) & (a_x s_1 - a_y c_1) & (p_x s_1 - p_y c_1) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Entry $(2,3) \rightsquigarrow a_z = 0$

Entry $(2,4) \rightsquigarrow p_z = 0$

$$\left.\begin{array}{l} \text{Entry } (2,1) \rightarrow n_z = s_2 \\ \text{Entry } (2,2) \rightsquigarrow o_z = c_2 \end{array}\right\} \tan\theta_2 = \frac{s_2}{c_2} = \frac{n_z}{o_z} \Rightarrow \theta_2 = \operatorname{atan2}(n_z, o_z)$$

Entry $(3,2) \rightarrow o_x s_1 - o_y c_1 = 0 \Rightarrow \tan\theta_1 = \frac{s_1}{c_1} = \frac{o_y}{o_x} \Rightarrow \theta_1 = \operatorname{atan2}(o_y, o_x)$

Entry $(1,3) \rightsquigarrow a_x c_1 + a_y s_1 = 0 \Rightarrow \tan\theta_1 = \frac{s_1}{c_1} = \frac{-a_x}{a_y} \Rightarrow \theta_1 = \operatorname{atan2}(-a_x, a_y$

$$\left.\begin{array}{l} \text{Entry } (1,4) \rightsquigarrow p_x c_1 + p_y s_1 = l_1 \\ \text{Entry } (3,4) \rightsquigarrow p_x s_1 - p_y c_1 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} p_y = l_1 s_1 \\ p_x = l_1 c_1 \end{array}$$

Given $\theta \longrightarrow$ find T (forward kinematics)

Given $T \longrightarrow$ find θ (inverse kinematics)

SOLVING KINEMATIC EQUATIONS 30.10.2002 / Wednesday

Euler Transform Solutions.

$$\text{Euler}(\phi, \theta, \psi) = T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x = \cos\phi \cdot \cos\theta \cdot \cos\psi - \sin\phi \cdot \sin\psi$

$n_y = \sin\phi \cdot \cos\theta \cos\psi + \cos\phi \cdot \sin\psi$

$n_z = -\sin\theta \cdot \cos\psi$

$o_x = -\cos\phi \cos\theta \sin\psi - \sin\phi \cos\psi$

$o_y = -\sin\phi \cdot \cos\theta \cdot \sin\psi + \cos\phi \cdot \cos\psi$

$o_z = \sin\theta \sin\psi$

$a_x = \cos\phi . \sin\theta$

$a_y = \sin\phi . \sin\theta$

$a_z = \cos\theta$

$\theta = \cos^{-1} a_z$

$\phi = \cos^{-1}\left(\dfrac{a_x}{\sin\theta}\right)$

$\psi = \cos^{-1}\left(-\dfrac{n_z}{\sin\theta}\right)$

EULER $= Rot(z,\phi) . Rot(y,\theta) . Rot(z,\psi) = T$

$\Rightarrow \quad Rot(y,\theta) . Rot(z,\psi) = Rot^{-1}(z,\phi) . T$

$$\begin{bmatrix} \cos\theta . \cos\psi & -\cos\theta . \sin\psi & \sin\theta & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ -\sin\theta . \cos\psi & \sin\theta . \sin\psi & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Entry (3, 4) $\quad P_z = 0$

Entry (2,4) $\quad -P_x . \sin\phi + P_y . \cos\phi = 0 \rightarrow \tan\phi = \dfrac{P_y}{P_x}$

Entry (1,4) $\quad P_x . \cos\phi + P_y . \sin\phi = 0 \rightarrow \tan\phi = -\dfrac{P_x}{P_y}$

$\Rightarrow P_x = 0 , \quad P_y = 0$

Entry (2,3) $\quad -a_x \sin\phi + a_y . \cos\phi = 0 \Rightarrow \tan\phi = \dfrac{a_y}{a_x} \Rightarrow \phi = \operatorname{atan2}(a_y , a_x)$

Entry (1,3) $\quad a_x . \cos\phi + a_y . \sin\phi = \sin\theta$

Entry (3,3) $\quad \cos\theta = a_z$

$\tan\theta = \dfrac{a_x \cos\phi + a_y \sin\phi}{a_z}$

$\Rightarrow \theta = \operatorname{atan2}\left(a_x \cos\phi + a_y \sin\phi , a_z\right)$

Entry (2,1) $\quad -n_x . \sin\phi + n_y . \cos\phi = \sin\psi$

Entry (2,2) $\quad -o_x . \sin\phi + o_y . \cos\phi = \cos\psi$

$\tan\psi = \dfrac{-n_x . \sin\phi + n_y . \cos\phi}{-o_x . \sin\phi + o_y . \cos\phi}$

$\Rightarrow \psi = \operatorname{atan2}\left(-n_x . \sin\phi + n_y . \cos\phi , -o_x . \sin\phi + o_y . \cos\phi\right)$

RPY Transform Solutions:

$$RPY(\phi,\theta,\psi) = T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta.\sin\psi & \sin\theta.\cos\psi & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ -\sin\theta & \cos\theta.\sin\psi & \cos\theta.\cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot(y,\theta).Rot(x,\psi)} = \underbrace{\begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot^{-1}(z,\phi)} . \underbrace{\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T}$$

Entry (3,4) $\quad \underline{p_z = 0}$

Entry (2,4) $\quad -p_x.\sin\phi + p_y.\cos\phi = 0$

Entry (1,4) $\quad p_x.\cos\phi + p_y.\sin\phi = 0$

$$\left.\begin{matrix} -p_x.\sin\phi + p_y.\cos\phi = 0 \\ p_x.\cos\phi + p_y.\sin\phi = 0 \end{matrix}\right\} \underline{p_x = 0}, \ \underline{p_y = 0}$$

Entry (2,1) $\quad -n_x.\sin\phi + n_y.\cos\phi = 0 \Rightarrow \tan\phi = \dfrac{n_y}{n_x} \Rightarrow \underline{\phi = \text{atan2}(n_y, n_x)}$

Entry (1,1) $\quad \cos\theta = n_x.\cos\phi + n_y.\sin\phi$

Entry (3,1) $\quad -\sin\theta = n_z$

$$\left.\begin{matrix} \cos\theta = n_x.\cos\phi + n_y.\sin\phi \\ -\sin\theta = n_z \end{matrix}\right\} \tan\theta = \frac{-n_z}{n_x.\cos\phi + n_y.\sin\phi}$$

$$\Rightarrow \theta = \text{atan2}(-n_z,\ n_x\cos\phi + n_y\sin\phi)$$

Entry (2,2) $\quad \cos\psi = -o_x.\sin\phi + o_y.\cos\phi$

Entry (2,3) $\quad -\sin\psi = -a_x.\sin\phi + a_y.\cos\phi$

$$\psi = \text{atan2}\big((a_x\sin\phi \ [illegible]\ a_y\cos\phi),\ (-o_x.\sin\phi + o_y.\cos$$

Spherical Transform Solutions.

$$Sph(\alpha,\beta,r) = T$$

$$Rot(z,\alpha).Rot(y,\beta).Transl(0,0,r) = T$$

$$\underbrace{\begin{bmatrix} \cos\beta & 0 & \sin\beta & r\sin\beta \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & r\cos\beta \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot(y,\beta).Transl(0,0,r)} = \underbrace{\begin{bmatrix} \cos\alpha & +\sin\alpha & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{Rot^{-1}(z,\alpha)} . \underbrace{\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{T}$$

Entry (2,4) $\quad 0 = -p_x \cdot \sin\alpha + p_y \cdot \cos\alpha \Rightarrow \tan\alpha = \dfrac{p_y}{p_x} \Rightarrow \underline{\alpha = \text{atan2}(p_y, p_x)}$

Entry (1,4) $\quad r \cdot \sin\beta = p_x \cdot \cos\alpha + p_y \cdot \sin\alpha$

Entry (3,4) $\quad r \cos\beta = p_z$

$$\left.\begin{aligned} r\sin\beta &= p_x\cdot\cos\alpha + p_y\cdot\sin\alpha \\ r\cos\beta &= p_z \end{aligned}\right\} \quad \underline{\beta = \text{atan2}\left((p_x \cdot \cos\alpha + p_y \cdot \sin\alpha), p_z\right)}$$

$$\left.\begin{aligned} r \sin^2\beta &= \sin\beta\,(p_x \cos\alpha + p_y \sin\alpha) \\ r\cos^2\beta &= \cos\beta \cdot p_z \end{aligned}\right\} \quad \underline{r = \cos\beta \cdot p_z + \sin\beta\,(p_x \cdot \cos\alpha + p_y \cdot \sin\alpha)}$$

Cylindrical Transform Solutions:

$\text{Cyl}(z, \alpha, r) = T$

$\text{Transl}(0,0,z) \cdot \text{Rot}(z,\alpha) \cdot \text{Transl}(r,0,0) = T$

$$\underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & r\cos\alpha \\ \sin\alpha & \cos\alpha & 0 & r\sin\alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rot}(z,\alpha)\ \text{Transl}(r,0,0)} = \underbrace{\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z - z \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Transl}^{-1}(0,0,z) \cdot T}$$

$n_z = 0 \quad , o_z = 0 \quad , a_z = 1 \quad , a_x = 0 \quad , a_y = 0$

Entry (3,4) $\quad 0 = p_z - z \Rightarrow \underline{z = p_z}$

Entry (1,4) $\quad \left.\begin{aligned} r \cdot \cos\alpha &= p_x \\ r \cdot \sin\alpha &= p_y \end{aligned}\right\} \tan\alpha = \dfrac{p_y}{p_x} \Rightarrow \underline{\alpha = \text{atan2}(p_y, p_x)}$

$$\begin{aligned} r \cdot \cos^2\alpha &= p_x \cdot \cos\alpha \\ r \cdot \sin^2\alpha &= p_y \cdot \sin\alpha \end{aligned} \Rightarrow r = p_x \cdot \cos\alpha + p_y \cdot \sin\alpha$$

A_5 A_6

$$\begin{bmatrix} 0.176 & \overset{0}{\cancel{0.663}} & \overset{-0.984}{\cancel{0}} & \overset{0}{\cancel{0}} \\ -0.984 & 0 & -0.176 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.747 & 0.663 & 0 & 0 \\ -0.663 & 0.747 & 0 & 0 \\ 0 & 0 & 1 & 0.2629 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.13552 & 0.116688 & -0.984 & -0.2586 \\ 0.735048 & 0.652391 & -0.176 & -0.0462704 \\ -0.663 & 0.747 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

DIFFERENTIAL RELATIONSHIPS

$$T = \begin{bmatrix} t_{11} & \cdots & t_{14} \\ \vdots & & \vdots \\ t_{41} & \cdots & t_{44} \end{bmatrix} \qquad dT = \begin{bmatrix} \frac{\partial t_{11}}{\partial x} & \cdots & \frac{\partial t_{14}}{\partial x} \\ \vdots & & \vdots \\ \frac{\partial t_{41}}{\partial x} & \cdots & \frac{\partial t_{44}}{\partial x} \end{bmatrix} dx$$

Differential Translation and Rotation

$$T + dT = \underbrace{\left[Transl(dx, dy, dz) Rot(k, d\theta) \right]}_{\text{differential changes are represented in base coordinates}} T$$

$$= T \left[\underbrace{Transl(dx, dy, dz) Rot(k, d\theta)}_{\text{differential changes are represented in its own coordinate reference frame}} \right]$$

C

$$dT = \left[Transl(dx, dy, dz) Rot(k, d\theta) - I \right] T$$

$$= T \underbrace{\left[Transl(dx, dy, dz) Rot(k, d\theta) - I \right]}_{{}^{T}\Delta}$$

differential translation and rotation transformation

$$dT = T \, {}^{T}\Delta$$

$$= \Delta T$$

587

(2)

(6-10-2002)

$$\mathrm{Transl}(\underline{d}) = \begin{bmatrix} 1 & & 0 & dx \\ & 1 & & dy \\ 0 & & 1 & dz \\ & & & 1 \end{bmatrix}$$

$$\underline{d} = i\,dx + j\,dy + k\,dz$$

$$\mathrm{Rot}(k, \theta) = \begin{bmatrix} k_x^2 \mathrm{Vers}\theta + \cos\theta & k_x k_y \mathrm{Vers}\theta - k_z \sin\theta & k_x k_z \mathrm{Vers}\theta + k_y \sin\theta & 0 \\ k_x k_y \mathrm{Vers}\theta + k_z \sin\theta & k_y^2 \mathrm{Vers}\theta + \cos\theta & k_y k_z \mathrm{Vers}\theta - k_x \sin\theta & 0 \\ k_x k_z \mathrm{Vers}\theta - k_y \sin\theta & k_y k_z \mathrm{Vers}\theta + k_x \sin\theta & k_z^2 \mathrm{Vers}\theta + \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(k, d\theta) = \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta & 0 \\ k_z d\theta & 1 & -k_x d\theta & 0 \\ -k_y d\theta & k_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lim_{d\theta \to 0} \sin d\theta \to d\theta$$

$$\lim_{d\theta \to 0} \cos d\theta \to 1$$

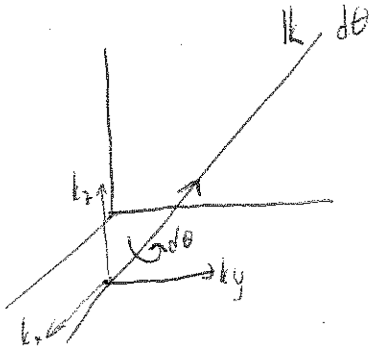
$$\lim_{d\theta \to 0} \mathrm{Vers}\, d\theta \to 0$$

$$\Delta = \mathrm{Transl}(\underline{d})\, \mathrm{Rot}(k, d\theta) - I$$

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -k_z d\theta & k_y d\theta & 0 \\ k_z d\theta & 1 & -k_x d\theta & 0 \\ -k_y d\theta & k_x d\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} I \end{bmatrix}$$

$$\Delta = \begin{bmatrix} 0 & -k_z d\theta & k_y d\theta & dx \\ k_z d\theta & 0 & -k_x d\theta & dy \\ -k_y d\theta & k_x d\theta & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{d}$



$$\mathrm{Rot}(x, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(x, \delta x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

587

3

6-10-2002

$$Rot(y,\delta y) = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad Rot(z,\delta z) = \begin{bmatrix} 1 & -\delta z & 0 & 0 \\ \delta z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(x,\delta x)\,Rot(y,\delta y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\delta x & 0 \\ 0 & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \delta y & 0 \\ 0 & 1 & 0 & 0 \\ -\delta y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \delta y & 0 \\ \cancel{\delta x \delta y} \to 0 & 1 & -\delta x & 0 \\ -\delta y & \delta x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

will be zero since too small quantity

$$Rot(x,\delta x)\,Rot(y,\delta y)\,Rot(z,\delta z) = \begin{bmatrix} 1 & \overset{-k_z d\theta}{-\delta z} & \overset{k_y d\theta}{\delta y} & 0 \\ \overset{k_z d\theta}{\delta z} & 1 & \overset{-k_x d\theta}{-\delta x} & 0 \\ \overset{-k_y d\theta}{-\delta y} & \overset{k_x d\theta}{\delta x} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{aligned} \delta x &= k_x d\theta \\ \delta y &= k_y d\theta \\ \delta z &= k_z d\theta \end{aligned}}$$

$$\Delta = \begin{bmatrix} 0 & -k_z d\theta & k_y d\theta & dx \\ k_z d\theta & 0 & -k_x d\theta & dy \\ -k_y d\theta & k_x d\theta & 0 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can interpret that

$Rot(k, d\theta)$ as $Rot(x,\delta x)\,Rot(y,\delta y)\,Rot(z,\delta z)$

$$\Delta = \begin{bmatrix} 0 & -\delta z & \delta y & dx \\ \delta z & 0 & -\delta x & dy \\ -\delta y & \delta x & 0 & dz \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{l}\underline{d} = i\,dx + j\,dy + k\,dz \\ \underline{\delta} = i\,\delta x + j\,\delta y + k\,\delta z\end{array}\right\}$$ refered as "differential motion vector" collectively

$$\underline{D} = \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix}$$