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$$A = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Differential Translation and Rotation Transformation (DTRT)

$d = i d_x + j d_y + k d_z$ (Differential translation vector)

$\delta = i\delta_x + j\delta_y + k\delta_z$ (Differential rotation vector)

$$D = \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$ Differential motion vector

$$A = \left[ \text{Transl}(d_x, d_y, d_z)\, \text{Rot}(k, d\theta) - I \right]$$

Example:

Given a coordinate frame A

$$A = \begin{array}{cccc} \text{x is along y} & \text{y is along z} & \text{z is along x} & \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the differential transformation corresponding to a differential translation vector of $d = 1i + 0j + 0.5k$ and differential rotation vector of $\delta = 0i + 0.1j + 0k$ made with respect to base coordinate?

Find also the location and orientation of the frame after the move?

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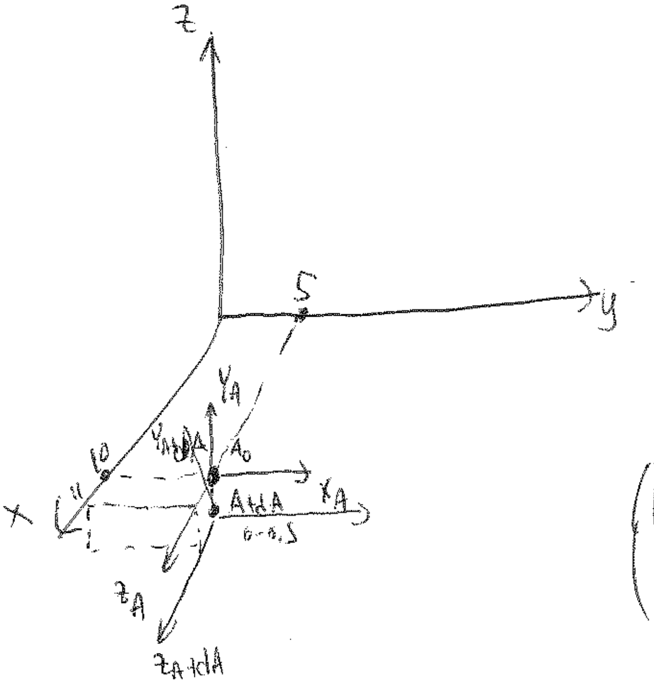
$$
{}^{0}\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}
\qquad
\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

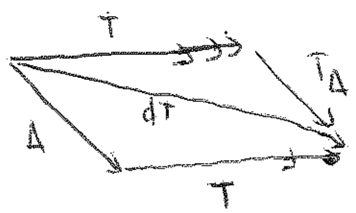
$$
dA = \Delta A = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$
w.r.t base coordinates

$$
A_{new} = A + dA = \begin{bmatrix} 0 & 0.1 & 1 & 11 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & -0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$




(look at the example for $Y_A, X_A, Z_A$ coordinates)

## Transforming Differential Changes DTR Between Coordinate frames




$$dT = \Delta T$$
$$= T\,{}^{T}\Delta$$

$$\boxed{\Delta T = T\,{}^{T}\Delta}$$

$$\boxed{\Delta = T\,{}^{T}\Delta\, T^{-1}}$$

$$\boxed{T^{-1}\,\Delta\, T = {}^{T}\Delta}$$

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$$^T\Delta = T^{-1}\Delta T = \begin{bmatrix} n_x & n_y & n_z & -p\cdot n \\ o_x & o_y & o_z & -p\cdot o \\ a_x & a_y & a_z & -p\cdot a \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} (\delta\times n)_x & (\delta\times o)_x & (\delta\times a)_x & ((\delta\times p)+d)_x \\ (\delta\times n)_y & (\delta\times o)_y & (\delta\times a)_y & ((\delta\times p)+d)_y \\ (\delta\times n)_z & (\delta\times o)_z & (\delta\times a)_z & ((\delta\times p)+d)_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} n\cdot(\delta\times n) & n\cdot(\delta\times o) & n(\delta\times a) & n((\delta\times p)+d) \\ o\cdot(\delta\times n) & o(\delta\times o) & o(\delta\times a) & o((\delta\times p)+d) \\ a(\delta\times n) & a(\delta\times o) & a(\delta\times a) & a((\delta\times p)+d) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a\cdot(b\times c) = -b(a\times c)$
$= b(c\times a)$

$a\cdot(a\times c) = 0$

$$^T\Delta = \begin{bmatrix} 0 & -\delta\cdot(n\times o) & \delta(a\times n) & \delta\cdot(p\times n)+d\cdot n \\ \delta(n\times o) & 0 & -\delta(o\times a) & \delta\cdot(p\times o)+d\cdot o \\ -\delta(a\times n) & \delta\cdot(o\times a) & 0 & \delta\cdot(p\times a)+d\cdot a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$n\times o = a$
$a\times n = o$
$o\times a = n$

$$^T\Delta = \begin{bmatrix} 0 & -\delta\cdot a & \delta\cdot o & \delta(p\times n)+d\cdot n \\ \delta\cdot a & 0 & \delta n & \delta(p\times o)+d\cdot o \\ -\delta\cdot o & \delta\cdot n & 0 & \delta(p\times a)+d\,a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$^T\Delta = \begin{bmatrix} 0 & -{}^T\delta_z & {}^T\delta_y & {}^Td_x \\ {}^T\delta_z & 0 & -{}^T\delta_x & {}^Td_y \\ -{}^T\delta_y & {}^T\delta_x & 0 & {}^Td_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

defined in base coordinate

$^T\delta_x = \delta \cdot n$ $\quad$ $^Td_x = \delta \cdot (p \times n) + d \cdot n$

$^T\delta_y = \delta \cdot o$ $\quad$ $^Td_y = \delta \cdot (p \times o) + d \cdot o$

$^T\delta_z = \delta \cdot a$ $\quad$ $^Td_z = \delta \cdot (p \times a) + d \cdot a$

$$\underbrace{\begin{bmatrix} ^Td_x \\ ^Td_y \\ ^Td_z \\ ^T\delta_x \\ ^T\delta_y \\ ^T\delta_z \end{bmatrix}}_{^TD} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_y & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \underbrace{\begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}}_{D}$$

Example:

Given the same coordinate frame $A$, the differential translation $\underline{d}$ and rotation $\delta$ of the previous example;

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} \underline{d} &= 1i + 0j + 0.5k \\ \underline{\delta} &= 0i + 0.1j + 0k \end{aligned}\right\} \text{w.r.t base coordinates}$$

What is the equivalent DTR transformation, $^A\Delta$ in coordinate frame $A$?

$dA = \Delta A = A\,^T\Delta$ $\qquad$ $\Delta A = A\,^A\Delta$

$A^{-1}\Delta A = {}^A\Delta$

$^Td_x = n \cdot [(\delta \times p) + d]$

$$\delta \times p = \begin{vmatrix} i & j & k \\ 0 & 0.1 & 0 \\ 10 & 5 & 0 \end{vmatrix} = 0i + 0j - 1k$$

$$[(\delta \times p) + d] = 1i + 0j - 0.5k$$

$^Ad_x = 0$ $\qquad$ $^A\delta_x = 0.1$ $\qquad$ $^A\underline{d} = 0i - 0.5j + 1k$

$^Ad_y = -0.5$ $\qquad$ $^A\delta_y = 0$ $\qquad$ $^A\delta = 0.1i + 0j + 0k$

$^Ad_z = 1$ $\qquad$ $^A\delta_z = 0$

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$$A_\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.3 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad dA = A\,{}^{A}\Delta = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= {}^{0}\Delta A$$

same with the one found in the previous example

$$\boxed{{}^{A}d_x = {}^{0}n \cdot \left[ ({}^{0}\delta \times {}^{0}p) + {}^{0}d \right]} \quad \boxed{{}^{0}d_x = {}^{A}n \left[ ({}^{A}\delta \times {}^{A}p) + {}^{A}d \right]} \qquad T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{T}d_x = n \cdot ((\delta \times p) + d) \qquad {}^{T}\delta_x = n \cdot \delta$$

$${}^{T}d_y = o \cdot ((\delta \times p) + d) \qquad {}^{T}\delta_y = o \cdot \delta$$

$${}^{T}d_z = a \cdot ((\delta \times p) + d) \qquad {}^{T}\delta_z = a \cdot \delta$$

$$\boxed{\Delta = T\,{}^{T}\Delta\,T^{-1}}$$

$$\boxed{{}^{T}\Delta = T^{-1}\,\Delta\,T}$$

$$\boxed{\Delta = (T^{-1})^{-1}\,{}^{T}\Delta\,T^{-1}}$$

$${}^{(T^{-1})}d_x = {}^{(T^{-1})}n \left( ({}^{T\Delta}\delta \times {}^{T^{-1}}p) + {}^{T\Delta}d \right) = {}^{0}d_x$$

$${}^{(T^{-1})}d_y = {}^{(T^{-1})}o \left( ({}^{T\Delta}\delta \times {}^{T^{-1}}p) + {}^{T\Delta}d \right) = {}^{0}d_y$$

$${}^{(T^{-1})}d_z = {}^{(T^{-1})}a \left( ({}^{T\Delta}\delta \times {}^{T^{-1}}p) + {}^{T\Delta}d \right) = {}^{0}d_z$$

$${}^{(T^{-1})}\delta_x = {}^{(T^{-1})}n \; {}^{T\Delta}\delta = {}^{0}\delta_x$$

$${}^{(T^{-1})}\delta_y = {}^{(T^{-1})}o \; {}^{T\Delta}\delta = {}^{0}\delta_y$$

$${}^{(T^{-1})}\delta_z = {}^{(T^{-1})}a \; {}^{T\Delta}\delta = {}^{0}\delta_z$$

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Example: Compute $\Delta$ in terms of $^{A}\Delta$ of the previous example

$$\Delta = (A^{-1})^{-1}\, {}^{A}\Delta\, (A^{-1})$$

$$A = \begin{array}{c} \begin{matrix} n & o & a & p \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad A^{-1} = \begin{array}{c} \begin{matrix} A^{-1}_n & A^{-1}_o & A^{-1}_a & A^{-1}_p \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$^{A}\Delta_{d} = [0\ \ -0.5\ \ 1]$

$^{A}\Delta_{\delta x} = 0.1$ $\quad {}^{A}\Delta_{\delta y} = 0$ $\quad {}^{A}\Delta_{\delta z} = 0$ $\quad {}^{A}\delta = [0.1\ \ 0\ \ 0]$

$\delta_x = n \cdot \delta = (0\ 0\ 1)\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0$

$\delta_y = o \cdot \delta = (1\ 0\ 0)\ 0.1$

$\delta_z = (0\ 1\ 0)\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0$

$$^{A}\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$^{A}\Delta_{\delta x} = 0.1$ $\qquad {}^{A}\Delta_{dx} = 0$

$^{A}\Delta_{\delta y} = 0$ $\qquad {}^{A}\Delta_{dy} = 0.5$

$^{A}\Delta_{\delta z} = 0$ $\qquad {}^{A}\Delta_{dz} = 1$

$$^{o}\delta_x = {}^{(A^{-1})}\delta_x = [0\ \ 0\ \ 1]\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$^{o}\delta_z = [0\ \ 1\ \ 0]\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$^{o}\delta_y = [1\ \ 0\ \ 0]\begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0.1$$

---

$$^{A}\Delta_{\delta x} \times {}^{A^{-1}}p = \begin{vmatrix} i & j & k \\ 0.1 & 0 & 0 \\ -5 & 0 & -10 \end{vmatrix} = -0.1(-10j) = j$$

$$\left[\left({}^{A}\Delta_{\delta x} \times {}^{A^{-1}}p\right) + {}^{A}\Delta_{d}\right] = 0.5j + k$$

$$^{o}d_x = [0\ \ 0\ \ 1]\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 1$$

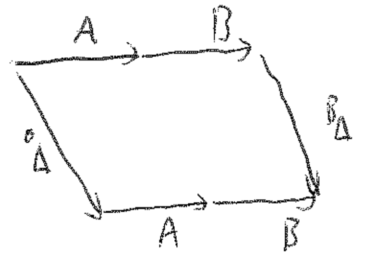
$$^{o}d_y = [1\ \ 0\ \ 0]\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 0$$

$$^{o}d_z = [0\ \ 1\ \ 0]\begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 0.5$$

$$A = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A B

$$\Delta A B = A B \, {}^{B}\Delta$$



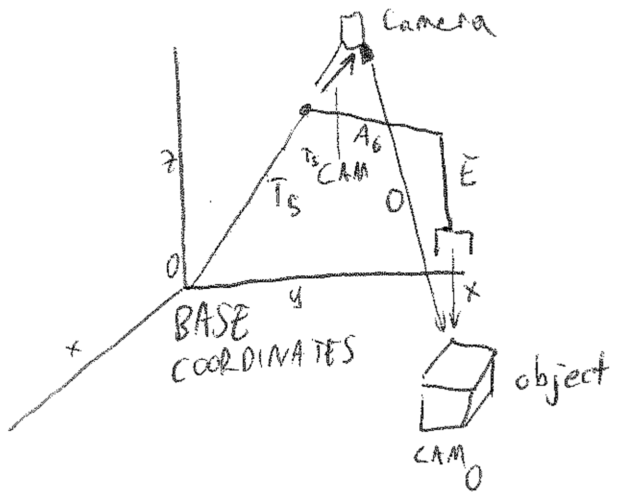

$${}^{A}\Delta B = {}^{A}B \, {}^{B}\Delta$$

$$(A\,{}^{A}\Delta) B = A \, {}^{A}B \, {}^{B}\Delta \qquad A \qquad A\,{}^{A}\Delta = \Delta A$$

$$\Delta A \; B = A \, {}^{A}B \, {}^{B}\Delta$$

$$\Delta = A B \, {}^{B}\Delta \, B^{-1} A^{-1} = \underbrace{(B^{-1}A^{-1})^{-1}}_{T^{-1}} \, {}^{B}\Delta \, \underbrace{(B^{-1}A^{-1})}_{T}$$

Ex: A camera is attached to link 5 of a manipulator as shown below




$${}^{T_5}CAM = CAM = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{Both w.r.t } T_5$$

$$A_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An object ${}^{CAM}O$ is observed and differential changes in ~~CAM~~ coordinates are given in order to bring the end-effector into contact with the object as

$${}^{CAM}d = -1i + 1j + 0k \qquad {}^{CAM}\delta = 0i + 0j + 0.1k$$

What are the required changes in $T_6$ coordinates?

showing that Δ is wrt base coordinates

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$${}^{R}_{0}\Delta = \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

→ Differential Translation and Rotation Transformation (DTRT)

$d = i.d_x + j.d_y + k.d_z$ Differential translation vector

$\delta = i.\delta_x + j.\delta_y + k.\delta_z$ Differential rotation vector

$$D = \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$ Differential motion vector

$$\Delta = \left[ \text{Transl}(d_x, d_y, d_z) . \text{Rot}(k, d\theta) - I \right]$$

Example: Given a coordinate frame A

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

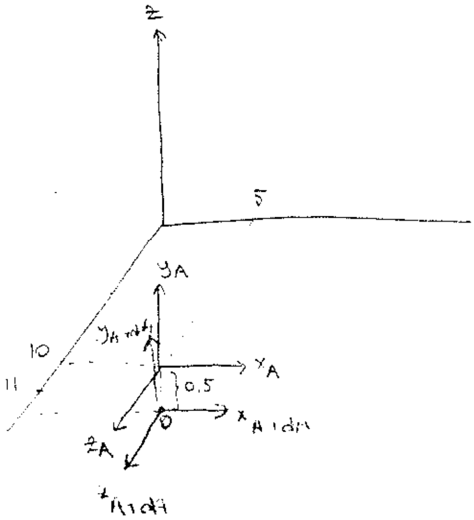
⇒ What is the differential transformation, dA corresponding to a differential translation vector of $d = 1i + 0j + 0.5k$ and differential rotation vector of $\delta = 0.i + 0.1j + 0.k$ made w.r.t. base coordinates?

Find also the location and orientation of the frame after the move.

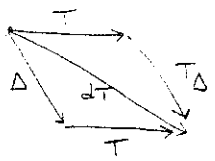
$$\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$dA = \Delta A = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ w.r.t. base coordinates.

$$A_{new} = A + dA = \begin{bmatrix} 0 & 0.1 & 1 & 11 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & -0.1 & -0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Transforming Differential Changes DTR Between Coordinate Frames.




$$dT = \Delta . T$$
$$= T . {}^{T}\Delta$$

identical

$$\Rightarrow \Delta T = T . {}^{T}\Delta$$

$$\Rightarrow \Delta = T \; {}^{T}\Delta . T^{-1}$$

or $\Rightarrow \boxed{{}^{T}\Delta = T^{-1} \Delta T}$

$${}^{T}\Delta = T^{-1} . \Delta . T$$

$$= \begin{bmatrix} n_x & n_y & n_z & -p.n \\ o_x & o_y & o_z & -p.o \\ a_x & a_y & a_z & -p.a \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 0 & -\delta_z & \delta_y & d_x \\ \delta_z & 0 & -\delta_x & d_y \\ -\delta_y & \delta_x & 0 & d_z \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow {}^{T}\Delta = \begin{bmatrix} n_x & n_y & n_z & -p.n \\ o_x & o_y & o_z & -p.o \\ a_x & a_y & a_z & -p.a \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} (\delta \times n)_x & (\delta \times o)_x & (\delta \times a)_x & ((\delta \times p)+d)_x \\ (\delta \times n)_y & (\delta \times o)_y & (\delta \times a)_y & ((\delta \times p)+d)_y \\ (\delta \times n)_z & (\delta \times o)_z & (\delta \times a)_z & ((\delta \times p)+d)_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow {}^{T}\Delta = \begin{bmatrix} n.(\delta \times n) & n.(\delta \times o) & n.(\delta \times a) & n.((\delta \times p)+d) \\ o.(\delta \times n) & o.(\delta \times o) & o.(\delta \times a) & o.((\delta \times p)+d) \\ a.(\delta \times n) & a.(\delta \times o) & a.(\delta \times a) & a.((\delta \times p)+d) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Note:</u> $a.(b \times c) = -b.(a \times c)$
$= b.(c \times a)$
$a.(a \times c) = 0$

$$\Rightarrow {}^{T}\Delta = \begin{bmatrix} 0 & -\delta(n \times o) & \delta(a \times n) & \delta(p \times n) + d.n \\ \delta(n \times o) & 0 & -\delta(o \times a) & \delta(p \times o) + d.o \\ -\delta(a \times n) & \delta(o \times a) & 0 & \delta(p \times a) + d.a \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>Note:</u> $n \times o = a$
$a \times n = o$
$o \times a = n$

$$\Rightarrow {}^{T}\Delta = \begin{bmatrix} 0 & -\delta.a & \delta.o & \delta.(p \times n) + d.n \\ \delta.a & 0 & -\delta.n & \delta.(p \times o) + d.o \\ -\delta.o & \delta.n & 0 & \delta.(p \times a) + d.a \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -{}^{T}\delta_z & {}^{T}\delta_y & {}^{T}d_x \\ {}^{T}\delta_z & 0 & -{}^{T}\delta_x & {}^{T}d_y \\ -{}^{T}\delta_y & {}^{T}\delta_x & 0 & {}^{T}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

${}^{T}\delta_x = \delta . n$ — defined in base coordinate

${}^{T}\delta_y = \delta . o$

${}^{T}\delta_z = \delta . a$

${}^{T}d_x = \delta . (p \times n) + d.n$

${}^{T}d_y = \delta . (p \times o) + d.o$

${}^{T}d_z = \delta . (p \times a) + d.a$

$$\underbrace{\begin{bmatrix} {}^T d_x \\ {}^T d_y \\ {}^T d_z \\ {}^T \delta_x \\ {}^T \delta_y \\ {}^T \delta_z \end{bmatrix}}_{{}^T D} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_y & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \cdot \underbrace{\begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}}_{D}$$

Example: Given the same coordinate frame A, the differential translation d, and rotation $\delta$ of the previous example;

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$d = 1.i + 0.j + 0.5k$ & $\delta = 0.i + 0.1j + 0.k$ $\Rightarrow$ wr.t base coordinates

$\Rightarrow$ What is equivalent DTR transformation ${}^A\Delta$ in coordinate frame A?

$${}^T d_x = n.\left[(\delta \times p) + d\right]$$

$$\delta \times p = \begin{vmatrix} i & j & k \\ 0 & 0.1 & 0 \\ 10 & 5 & 0 \end{vmatrix} = 0.i + 0j - 1.k$$

$$\left[(\delta \times p) + d\right] = 1.i + 0.j - 0.5k$$

${}^A d_x = 0$ ${}^A\delta_x = 0.1$

${}^A d_y = -0.5$ ${}^A\delta_y = 0$

${}^A d_z = 1$ ${}^A\delta_z = 0$

$${}^A d = 0i - 0.5j + 1.k \,, \quad {}^A\delta = 0.1i + 0.j + 0.k$$

$${}^A\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow dA = A \cdot {}^A\Delta = \begin{bmatrix} 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= {}^0\Delta . A$$

$\rightarrow$ Same with the one found in the previous example

Note that ${}^A d_x = {}^0 n . \left[({}^0\delta \times {}^0 p) + {}^0 d\right]$

$${}^0 d_x = {}^A n . \left[({}^A\delta \times {}^A p) + {}^A d\right]$$

$$^{T}d_x = n \cdot ((\delta \times p) + d) \qquad ^{T}\delta_x = n \cdot \delta$$

$$^{T}d_y = o \cdot ((\delta \times p) + d) \qquad ^{T}\delta_y = o \cdot \delta$$

$$^{T}d_z = a \cdot ((\delta \times p) + d) \qquad ^{T}\delta_z = a \cdot \delta$$

$$\Delta = T \cdot {}^{T}\Delta \cdot T^{-1}$$

$$^{T}\Delta = T^{-1} \cdot \Delta \cdot T$$ we're trying to write $\Delta$ in the format of $^{T}\Delta$

$$\Delta = (T^{-1})^{-1} \cdot {}^{T}\Delta \, (T^{-1})$$

$$(T^{-1})_{d_x} = {}^{T^{-1}}n \cdot \left( \left( {}^{T\Delta}\delta \times {}^{T^{-1}}p \right) + {}^{T\Delta}d \right) = {}^{0}d_x$$

$$(T^{-1})_{d_y} = {}^{T^{-1}}o \cdot \left( \left( {}^{T\Delta}\delta \times {}^{T^{-1}}p \right) + {}^{T\Delta}d \right) = {}^{0}d_y$$

$$(T^{-1})_{d_z} = {}^{T^{-1}}a \cdot \left( \left( {}^{T\Delta}\delta \times {}^{T^{-1}}p \right) + {}^{T\Delta}d \right) = {}^{0}d_z$$

Note that;

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} {}^{T^{-1}}n & {}^{T^{-1}}o & {}^{T^{-1}}a & {}^{T^{-1}}p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T^{-1})_{\delta_x} = {}^{T^{-1}}n \cdot {}^{T\Delta}\delta = {}^{0}\delta_x$$

$$(T^{-1})_{\delta_y} = {}^{T^{-1}}o \cdot {}^{T\Delta}\delta = {}^{0}\delta_y$$

$$(T^{-1})_{\delta_z} = {}^{T^{-1}}a \cdot {}^{T\Delta}\delta = {}^{0}\delta_z$$

Example: Compute $\Delta$ in terms of $^{A}\Delta$ of the previous example.

$$\Delta = (A^{-1})^{-1} \cdot {}^{A}\Delta \cdot (A^{-1})$$

$$A = \begin{array}{c} \begin{array}{cccc} n & o & a & p \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} , \qquad A^{-1} = \begin{array}{c} \begin{array}{cccc} {}^{A^{-1}}n & {}^{A^{-1}}o & {}^{A^{-1}}a & {}^{A^{-1}}p \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$^{A}\Delta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & -0.5 \\ 0 & 0.1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$^{A\Delta}\delta_x = 0.1 \qquad ^{A\Delta}d_x = 0$$

$$^{A\Delta}\delta_y = 0 \qquad ^{A\Delta}d_y = -0.5$$

$$^{A\Delta}\delta_z = 0 \qquad ^{A\Delta}d_z = 1$$

$$^{0}\delta_x = {}^{A^{-1}}\delta_x = [0 \;\; 0 \;\; 1] \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$$^{0}\delta_y = {}^{A^{-1}}\delta_y = [1 \;\; 0 \;\; 0] \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0.1$$

$$^{0}\delta_z = {}^{A^{-1}}\delta_z = [0 \;\; 1 \;\; 0] \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = 0.$$

$${}^{A}\Delta\delta \times {}^{A^{-1}}p = \begin{vmatrix} i & j & k \\ 0.1 & 0 & 0 \\ -5 & 0 & -10 \end{vmatrix} = -0.1(-10j) = j$$

$$\left({}^{A}\Delta\delta \times {}^{A^{-1}}p\right) + {}^{A}\Delta d = 0.5j + k$$

$${}^{0}d_x = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 1$$

$${}^{0}d_y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 0$$

$${}^{0}d_z = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix} = 0.5$$

$$\Delta A = A^{T}\Delta$$

$$\Delta = A^{T}\Delta A^{-1}$$

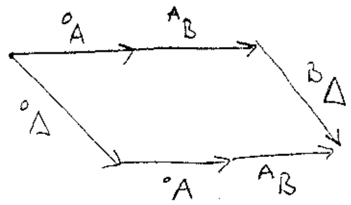
$$\Delta = \begin{bmatrix} 0 & 0 & 0.1 & 1 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

---

Generalization:

A B

$$\Delta A B = A . B\ {}^{B}\Delta$$




$${}^{A}\Delta B = {}^{A}B . {}^{B}\Delta$$

$$A . {}^{A}\Delta = \Delta . A$$

$$(A . {}^{A}\Delta)\, B = A\ {}^{A}B . {}^{B}\Delta$$

$$\Delta . A . B = A . {}^{A}B . {}^{B}\Delta$$

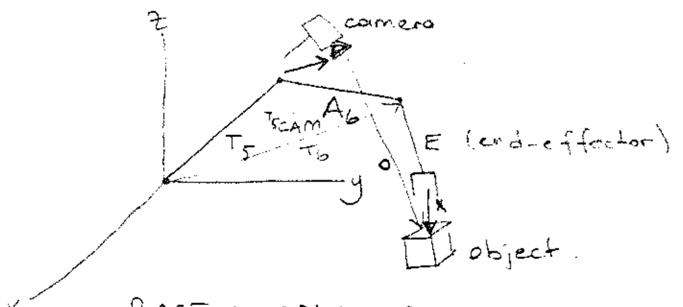
$$\Delta = A . B . {}^{B}\Delta\, B^{-1} . A^{-1}$$

$$\Delta = \underbrace{(B^{-1} . A^{-1})^{-1}}_{T} . {}^{B}\Delta\, \underbrace{(B^{-1} . A^{-1})}_{T^{-1}}$$

$$= T . {}^{B}\Delta . T^{-1}$$

Bir daha incele

Example. A camera is attached to link 5 of a manipulator as shown below:




BASE COORDINATES

$^{T_5}CAM$ = camera transformation wrt $T_5$

o = the object representation wrt camera

x = the object representation wrt end-effec.

short notation

$$^{T_5}CAM = CAM = \begin{bmatrix} 0 & 0 & -1 & 5 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \& \quad \underset{^{T_5}A_6}{A_6} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Bigg\} \text{both w.r.t } T_5$$

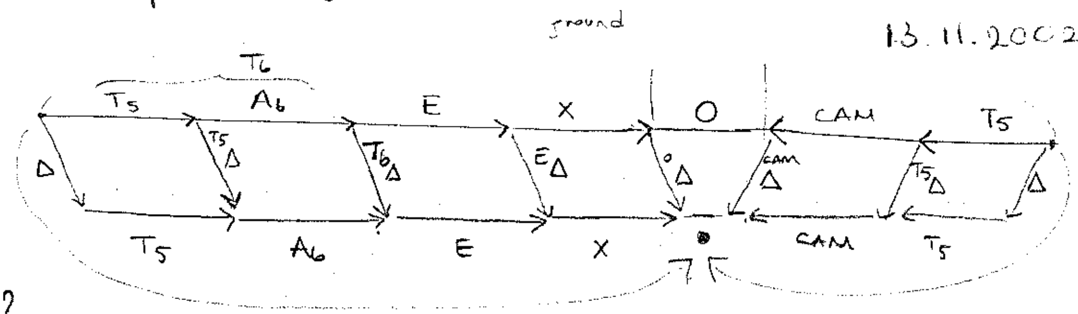
An object $^{CAM}o$ is observed and differential changes in CAM coordinates are given in order to bring the end-effector into contact with the object as

$$^{CAM}d = -1i + 1j + 0k$$

$$^{CAM}\delta = 0i + 0j + 0.1k$$

What are the required changes in $T_6$ coordinates?

13.11.2002 / Wednesday




$^{CAM}\underline{d}$

$^{CAM}\underline{\delta}$

$^{T_6}\Delta = ?$

$$\Delta T_5 A_6 = T_5 A_6 {}^{T_6}\Delta \Rightarrow \boxed{\Delta = T_5 . A_6 . {}^{T_6}\Delta \; A_6^{-1} \; T_5^{-1}}$$

$$\Delta T_5 \, CAM = T_5 \, CAM \, {}^{CAM}\Delta \Rightarrow \boxed{\Delta = T_5 \, CAM \, {}^{CAM}\Delta \, CAM^{-1} \, T_5^{-1}}$$

$$\cancel{T_5} A_6 {}^{T_6}\Delta \, A_6^{-1} \cancel{T_5^{-1}} = \cancel{T_5} \, CAM \, {}^{CAM}\Delta \, CAM^{-1} \cancel{T_5^{-1}} \Rightarrow A_6 {}^{T_6}\Delta \, A_6^{-1} = CAM \, {}^{CAM}\Delta \, CAM^{-1}$$

$$\Rightarrow \underbrace{A_6^{-1} . A_6}_{I} {}^{T_6}\Delta \underbrace{A_6^{-1} \, A_6}_{1} = A_6^{-1} \, CAM \, {}^{CAM}\Delta \, CAM^{-1} \, A_6$$

Let's define $T = CAM^{-1} A_6$

$$\Rightarrow {}^{T_6}\Delta = A_6^{-1} . CAM . {}^{CAM}\Delta \, CAM^{-1} . A_6$$

$$\Rightarrow {}^{T_6}\Delta = (CAM^{-1} . A_6)^{-1} \, {}^{CAM}\Delta \, (CAM^{-1} A_6) \Rightarrow \boxed{{}^{T_6}\Delta = T^{-1} . {}^{CAM}\Delta . T}$$

$$CAM^{-1} = \begin{bmatrix} 0 & 0 & -1 & 10 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

13.11. ②

$$\underbrace{CAM^{-1} \cdot A_6}_{T} = \begin{bmatrix} 0 & 0 & -1 & 10 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{array}{c} \begin{matrix} n & o & a & p \\ \downarrow & \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix} 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

${}^{CAM}d$
${}^{CAM}\delta$

$$^{T_6}\Delta = \begin{bmatrix} 0 & -{}^{T_6}\delta_z & {}^{T_6}\delta_y & {}^{T_6}d_x \\ {}^{T_6}\delta_z & 0 & -{}^{T_6}\delta_x & {}^{T_6}d_y \\ -{}^{T_6}\delta_y & {}^{T_6}\delta_x & 0 & {}^{T_6}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$^{T_6}d_x = {}^{T}\underline{n} \left( \left( {}^{CAM}\underline{\delta} \times {}^{T}\underline{p} \right) + {}^{CAM}\underline{d} \right)$$

$T \triangleq (CAM^{-1} A_6)$

$$^{T_6}\delta_x = {}^{T}\underline{n} \cdot {}^{CAM}\underline{\delta}$$

$$^{T_6}\delta_y = {}^{T}\underline{o} \cdot {}^{CAM}\underline{\delta}$$

$$^{T_6}\delta_z = {}^{T}\underline{a} \cdot {}^{CAM}\underline{\delta}$$

$$^{T_6}d_y = {}^{T}\underline{o} \cdot \left( \left( {}^{CAM}\underline{\delta} \times {}^{T}\underline{p} \right) + {}^{CAM}\underline{d} \right)$$

$$^{T_6}d_z = {}^{T}\underline{a} \cdot \left( \left( {}^{CAM}\delta \times {}^{T}\underline{p} \right) + {}^{CAM}\underline{d} \right)$$

$$\left( {}^{CAM}\underline{\delta} \times {}^{T}\underline{p} \right) = \begin{vmatrix} i & j & k \\ 0 & 0 & 0.1 \\ 2 & 0 & 5 \end{vmatrix} = 0.i + 0.2j + 0.k$$

$$\left[ \left( {}^{CAM}\underline{\delta} \times {}^{T}\underline{p} \right) + {}^{CAM}\underline{d} \right] = -1i + 1.2j + 0k$$

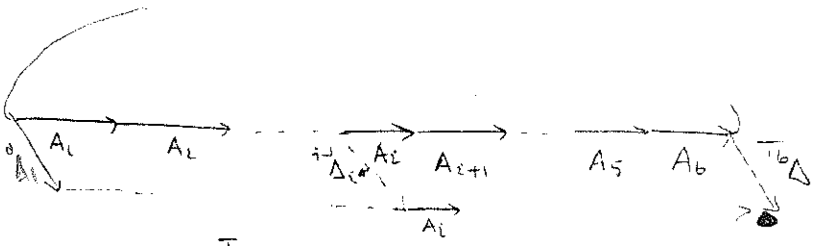
$$\left.\begin{aligned} {}^{T_6}d_x &= \begin{bmatrix} 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1.2 \\ 0 \end{bmatrix} = -\cancel{1}.2 \\ {}^{T_6}d_y &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1.2 \\ 0 \end{bmatrix} = 0 \\ {}^{T_6}d_z &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1.2 \\ 0 \end{bmatrix} = 1 \end{aligned}\right\} \quad {}^{T_6}\underline{d} = -\cancel{1}.2i + 0.j + 1.k$$

yanlışı buldun

$$^{T_6}\delta = 0i + 0.1j + 0.k$$

$$\Rightarrow \quad {}^{T_6}\Delta = \begin{bmatrix} 0 & 0 & 0.1 & -\cancel{1}.2 \\ 0 & 0 & 0 & 0 \\ -0.1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

MANIPULATOR JACOBIAN




$$dT_6 = T_6 \, {}^{T_6}\Delta \, dq_i$$

$$= A_1 \cdot A_2 \cdots A_{i-1} \, {}^{i-1}\Delta_i \cdot A_i \cdots A_5 A_6 \, dq_i$$

$$T_6^{-1} \cdot T_6 \cdot {}^{T_6}\Delta \cdot dq_i = T_6^{-1} A_1 \cdots A_{i-1} \, {}^{i-1}\Delta_i \, A_i \cdots A_5 A_6 \, dq_i$$

$${}^{T_6}\Delta = \underbrace{\left(A_6^{-1} \cdot A_5^{-1} \cdots A_i^{-1}\right)}_{T^{-1}} {}^{i-1}\Delta_i \cdot \underbrace{\left(A_i \cdots A_6\right)}_{T}$$

$${}^{T_6}d_x = {}^{T}\underline{n} \cdot \left(\left({}^{i-1\Delta_i}\underline{\delta} \times {}^{T}\underline{p}\right) + {}^{i-1\Delta_i}\underline{d}\right) \qquad {}^{T_6}\delta_x = {}^{T}\underline{n} \cdot {}^{i-1\Delta_i}\underline{\delta}$$

$T \triangleq A_i \cdots A_6$

$${}^{T_6}d_y = {}^{T}\underline{o} \left(\left({}^{i-1\Delta_i}\underline{\delta} \times {}^{T}\underline{p}\right) + {}^{i-1\Delta_i}\underline{d}\right) \qquad {}^{T_6}\delta_y = {}^{T}\underline{o} \cdot {}^{i-1\Delta_i}\underline{\delta}$$

$${}^{T_6}d_z = {}^{T}\underline{a} \left(\left({}^{i-1\Delta_i}\underline{\delta} \times {}^{T}\underline{p}\right) + {}^{i-1\Delta_i}\underline{d}\right) \qquad {}^{T_6}\delta_z = {}^{T}\underline{a} \cdot {}^{i-1\Delta_i}\underline{\delta}$$

If the joint is revolute joint (ith joint) then ${}^{i-1}\Delta_i$ has $\underline{\underline{no}}$ ${}^{i-1\Delta_i}d$ parameters: ${}^{i-1\Delta_i}d_x = 0$, ${}^{i-1\Delta_i}d_y = 0$, ${}^{i-1\Delta_i}d_z = 0$

If the joint is prismatic joint (ith joint) then ${}^{i-1}\Delta_i$ has $\underline{\underline{no}}$ ${}^{i-1\Delta_i}\delta$ parameters: ${}^{i-1\Delta_i}\delta_x = 0$, ${}^{i-1\Delta_i}\delta_y = 0$, ${}^{i-1\Delta_i}\delta_z = 0$

revolute

translational diff. motion resulting in the $T_6$ coord. frame

$${}^{T_6}\underline{d_i} = \left(-n_x \cdot p_y + n_y \cdot p_x\right) i + \left(-o_x \cdot p_y + o_y \cdot p_x\right) j + \left(-a_x \cdot p_y + a_y \cdot p_x\right) k$$

indicating a motion at the ith joint

$${}^{T_6}\delta_i = n_z \cdot i + o_z \cdot j + a_z \cdot k \qquad \delta_i = 0i + 0j + 1k$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ p_x & p_y & p_z \end{vmatrix} \to i \, (-p_y) + j \, p_x$$

If the joint is prismatic $\underline{\delta_i} = 0$, $\underline{d_i} = 0i + 0j + 1k$

$${}^{T_6}d_i = n_z i + o_z j + a_z k$$

$${}^{T_6}\delta_i = 0$$

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If joint is revolute, then $\underline{d_i} = 0$

$${}^{T_6}d_{ix} = n.(\delta \times p) \qquad {}^{T_6}\delta_{ix} = n.\delta$$

$${}^{T_6}d_{iy} = o.(\delta \times p) \qquad {}^{T_6}\delta_{iy} = o.\delta$$

$${}^{T_6}d_{iz} = a.(\delta \times p) \qquad {}^{T_6}\delta_{iz} = a.\delta$$

If z = axis of rotation:

$${}^{T_6}\underline{d_i} = \underbrace{(-n_x \cdot p_y + n_y \cdot p_x)}_{{}^{T_6}d_{ix}} . i + \underbrace{(-o_x \cdot p_y + o_y \cdot p_x)}_{{}^{T_6}d_{iy}} . j + \underbrace{(-a_x \cdot p_y + a_y \cdot p_x)}_{{}^{T_6}d_{iz}} . k$$

$${}^{T_6}\underline{\delta_i} = n_z . i + o_z . j + a_z . k$$

If the joint is prismatic $\underline{\delta_i} = 0$, $\underline{d_i} = 0.i + 0.j + 1.k$

$${}^{T_6}\underline{d_i} = n_z . i + o_z . j + a_z . k$$

$${}^{T_6}\underline{\delta_i} = 0$$

$${}^{T_6}\begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} = {}^{T_6}\underbrace{\begin{bmatrix} d_{1x} & d_{2x} & d_{3x} & d_{4x} & d_{5x} & d_{6x} \\ d_{1y} & d_{2y} & d_{3y} & d_{4y} & d_{5y} & d_{6y} \\ d_{1z} & d_{2z} & d_{3z} & d_{4z} & d_{5z} & d_{6z} \\ \delta_{1x} & \delta_{2x} & \delta_{3x} & \delta_{4x} & \delta_{5x} & \delta_{6x} \\ \delta_{1y} & \delta_{2y} & \delta_{3y} & \delta_{4y} & \delta_{5y} & \delta_{6y} \\ \delta_{1z} & \delta_{2z} & \delta_{3z} & \delta_{4z} & \delta_{5z} & \delta_{6z} \end{bmatrix}}_{\text{JACOBIAN } (J)} \cdot \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \\ dq_4 \\ dq_5 \\ dq_6 \end{bmatrix} \Rightarrow {}^{T_6}\begin{bmatrix} d \\ \delta \end{bmatrix} = {}^{T_6}\begin{bmatrix} J \end{bmatrix} \cdot \begin{bmatrix} dq \end{bmatrix}$$

# DYNAMICS

$L = K - P$ (K: Kinetic Energy, P = Potential Energy)

$\nearrow$ Lagrangian

$$F_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

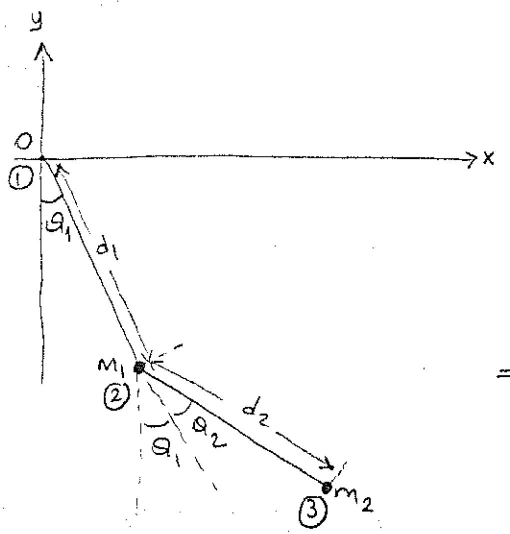
joint number

Joint force (if prismatic) or Joint torque (if revolute)

joint velocity

Joint variable

Example:




$g$ = gravitational field 9.8 m/sec$^2$

$\Rightarrow$ Two-link planar manipulator

The kinetic energy of mass $m_1$

$$K_1 = \frac{1}{2} m_1 . d_1^2 . \dot{\theta}_1^2$$

The potential energy of mass $m_1$

$$P_1 = -m_1 . g . d_1 . \cos\theta_1$$

$$v_2 = \sqrt{\frac{dx_2}{dt}^2 + \frac{dy_1}{dt}^2}$$

$$\boxed{v_1 = d_1 . \dot{\theta}_1}$$

For the second mass the positional expressions;

$$x_2 = d_1 . \sin\theta_1 + d_2 . \sin(\theta_1 + \theta_2)$$

$$y_2 = -d_1 . \cos\theta_1 - d_2 . \cos(\theta_1 + \theta_2)$$

---

$$\dot{x}_2 = d_1 . \cos\theta_1 . \dot{\theta}_1 + d_2 . \cos(\theta_1 + \theta_2) . (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\dot{y}_2 = d_1 . \sin\theta_1 . \dot{\theta}_1 + d_2 . \sin(\theta_1 + \theta_2) . (\dot{\theta}_1 + \dot{\theta}_2)$$

$$\Rightarrow \dot{x}_2^2 + \dot{y}_2^2 = v_2^2 = d_1^2 . \dot{\theta}_1^2 . \cos^2\theta_1 + 2 d_1 . d_2 . \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) . \cos\theta_1 . \cos(\theta_1 + \theta_2) + d_2^2 . (\dot{\theta}_1 + \dot{\theta}_2)^2 . \underbrace{\cos^2_{12}}_{\cos^2(\theta_1+\theta_2)}$$

$$+ d_1^2 . \dot{\theta}_1^2 . \underbrace{\sin^2_1}_{\sin^2\theta_1} + 2 d_1 d_2 . \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin_1 . \underbrace{\sin_{12}}_{\sin^2(\theta_1+\theta_2)} + d_2^2 (\dot{\theta}_1 + \dot{\theta}_2) . \sin^2_{12}$$

$$= d_1^2 . \dot{\theta}_1^2 + d_2^2 . (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2 d_1 d_2 \cos\theta_2 \, \dot{\theta}_1 . (\dot{\theta}_1 + \dot{\theta}_2)$$

Hence,

$$K_2 = \frac{1}{2} m_2 . v_2^2 = \frac{1}{2} m_2 d_1^2 \dot{\theta}_1^2 + m_2 d_1 d_2 \cos_2 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2} m_2 . d_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$P_2 = m_2 . g . y_2 = -m_2 . g . d_1 . \cos_1 - m_2 . g . d_2 . \cos_{12}$$

$$L = K - P = (K_1 + K_2) - (P_1 + P_2)$$

$$= \frac{1}{2} (m_1 + m_2) d_1^2 . \dot{\theta}_1^2 + \frac{1}{2} m_2 . d_2^2 . (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 . d_1 . d_2 . \cos_2 . \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) + (m_1 + m_2) g . d_1 \cos_1$$

$$+ m_2 . g . d_2 . \cos_{12}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)\cdot d_1^2\cdot\dot{\theta}_1 + m_2\, d_2^2\,(\dot{\theta}_1+\dot{\theta}_2) + m_2\cdot d_1\cdot d_2\cdot \cos_2\,(2\dot{\theta}_1+\dot{\theta}_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left[(m_1+m_2)\cdot d_1^2 + m_2\cdot d_2^2 + 2m_2\, d_1\cdot d_2\cdot\cos_2\right]\cdot\ddot{\theta}_1 + \left[m_2\cdot d_2^2 + m_2\cdot d_1\cdot d_2\cdot\cos_2\right]\cdot\ddot{\theta}_2$$
$$- m_2\cdot d_1\cdot d_2\cdot \sin_2\,\dot{\theta}_2\,(2\dot{\theta}_1+\dot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1+m_2)\cdot g\cdot d_1\cdot\sin_1 - m_2\cdot g\cdot d_2\cdot\sin_{12}$$

$$T_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1}$$

$$T_1 = \left[(m_1+m_2)\, d_1^2 + m_2 d_2^2 + 2m_2\, d_1 d_2\cdot\cos_2\right]\ddot{\theta}_1 + \left[m_2\cdot d_2^2 + m_2\cdot d_1 d_2\cdot\cos_2\right]\ddot{\theta}_2$$
$$- m_2\cdot d_1 d_2\cdot\sin_2\,\dot{\theta}_2\,(2\dot{\theta}_1+\dot{\theta}_2) + (m_1+m_2)\cdot g\cdot d_1\cdot\sin_1 + m_2\cdot g\cdot d_2\cdot\sin_{12}$$

For the joint 2

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2\cdot d_2^2\,(\dot{\theta}_1+\dot{\theta}_2) + m_2\cdot d_1\cdot d_2\cdot\cos\theta_2\cdot\dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2\, d_2^2\,(\ddot{\theta}_1+\ddot{\theta}_2) + m_2\cdot d_1\cdot d_2\cdot\cos_2\cdot\ddot{\theta}_1 - m_2\cdot d_1\cdot d_2\cdot\sin_2\cdot\dot{\theta}_1\cdot\dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2\cdot d_1\cdot d_2\,\sin_2\,\dot{\theta}_1\cdot(\dot{\theta}_1+\dot{\theta}_2) - m_2\cdot g\cdot d_2\cdot\sin_{12}$$

$$T_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$T_2 = m_2\, d_2^2\,(\ddot{\theta}_1+\ddot{\theta}_2) + m_2\, d_1\, d_2\cdot\cos_2\cdot\ddot{\theta}_1 - m_2\cdot d_1\cdot d_2\cdot\sin_2\,\dot{\theta}_1\,\dot{\theta}_2 + m_2\cdot d_1 d_2\cdot\sin_2\,\dot{\theta}_1\,(\dot{\theta}_1+\dot{\theta}_2)$$
$$+ m_2\cdot g\cdot d_2\cdot\sin_{12}$$

$$T_1 = \underbrace{D_{11}}_{\text{Effective Inertia}}\cdot\ddot{\theta}_1 + \underbrace{D_{12}^{*}}_{\text{Coupling Inertia}}\cdot\ddot{\theta}_2 + \underbrace{D_{111}\cdot\dot{\theta}_1^2 + D_{122}\cdot\dot{\theta}_2^2}_{\text{Centipetal forces}} + \underbrace{D_{112}\cdot\dot{\theta}_1\,\dot{\theta}_2 + D_{121}\cdot\dot{\theta}_2\cdot\dot{\theta}_1}_{\text{Coriolis forces}} + \underbrace{D_1}_{\text{Gravity force}}$$

$$T_2 = D_{21}^{*}\,\ddot{\theta}_1 + D_{22}\cdot\ddot{\theta}_2 + D_{211}\cdot\dot{\theta}_1^2 + D_{222}\cdot\dot{\theta}_2^2 + D_{212}\cdot\dot{\theta}_1\cdot\dot{\theta}_2 + D_{221}\cdot\dot{\theta}_2\,\dot{\theta}_1 + D_2$$

Effective Inertias:

$$D_{1_1} = \left[ (m_1+m_2).d_1^2 + m_2.d_2^2 + 2.m_2.d_1 d_2 . \cos_2 \right]$$

$$D_{2_2} = m_2 . d_2^2$$

Coupling Inertias: $D_{1_2} = D_{2_1} = m_2 . d_2^2 + m_2 . d_1 . d_2 . \cos_2$

Centipetal forces:

$D_{1_{11}} = 0$ $\qquad$ $D_{2_{11}} = -m_2 . d_1 . d_2 . \sin 2$ (Sanırım pozitif olacak)

$D_{1_{22}} = -m_2 . d_1 . d_2 . \sin 2$ $\qquad$ $D_{2_{22}} = 0$

Coriolis (Velocity) Force Coefficients:

$$D_{1_{12}} = D_{1_{21}} = -m_2 . d_1 . d_2 . \sin 2$$

$$D_{2_{12}} = D_{2_{21}} = -m_2 . d_1 . d_2 . \sin_2$$ $>0$ olmalı bence

Gravity Terms:

$$D_1 = (m_1+m_2) . g . d_1 . \sin_1 + m_2 . g . d_2 . \sin_{12}$$

$$D_2 = m_2 . g . d_2 . \sin_{12}$$

Let the manipulator be at rest (i.e. $\theta_1 = \theta_2 = 0$, and the speed is zero $\dot\theta_1 = \dot\theta_2 = 0$) in a gravity free environment (i.e. $D_1 = D_2 = 0$) with

i. Joint 2 is locked $\ddot\theta_2 = 0$

ii. Joint 2 is free $T_2 = 0$

i. $T_1 = D_{1_1} . \ddot\theta_1$ $\qquad$ $T_2 = D_{1_2} . \ddot\theta_1$

ii. $T_2 = 0 = D_{2_1} . \ddot\theta_1 + D_{2_2} . \ddot\theta_2$

$$\ddot\theta_2 = -\frac{D_{21}}{D_{2_2}} . \ddot\theta_1$$

$$T_1 = \left[ D_{1_1} - \frac{D_{1_2} D_{21}}{D_{2_2}} \right] \ddot\theta$$

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In summary of the method to be followed:

i. Choose the generalized coordinates $q_i$ $i=1,\dots,n$ as joint variables such as $\theta_i$ or $d_i$

ii. Obtain kinetic or potential energy expressions for the manipulator as;

$$K = \sum_{i=1}^{n} k_i \quad \text{and} \quad P = \sum_{i=1}^{n} p_i$$

yielding the Lagrangian, $L$ such that

$$L = K - P$$

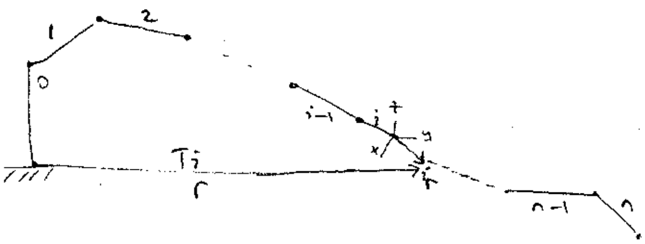
iii. Find the force or torque expressions depending on the type of the joint whether it is revolute or prismatic, respectively by using the Euler–Lagrange equation;

$$F_i \ (\text{or } \tau_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \qquad i=1,\dots,n$$

Velocity of a point on the Manipulator:

Given a point ${}^i r$ w.r.t link $i$ coordinate frame, its position in base coordinates is given by

$$r = T_i \cdot {}^i r \qquad \text{where } T_i = A_1 . A_2 \dots A_{i-1} . A_i$$




$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}\left(T_i \cdot {}^i r\right) = \frac{d}{dt}(T_i) \cdot {}^i r = \Big(\sum \frac{\partial T_i}{\partial q_j} \cdot \underbrace{\dot{q}_j}_{\frac{d\theta_j}{dt}}\Big) \cdot {}^i r$$

Note that $\underbrace{\begin{bmatrix} \cos\theta_j & \sin\theta_j & 0 & 0 \\ \sin\theta_j & \cos\theta_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{is depend of } \theta_j(t)}$ so

$$\frac{\partial T_i}{\partial q_j} = T_i \cdot {}^{T_i}\Delta$$

Note: ${}^{T_i}\Delta = \begin{bmatrix} 0 & -{}^{T_i}\delta_z & {}^{T_i}\delta_y & {}^{T_i}d_x \\ {}^{T_i}\delta_z & 0 & -{}^{T_i}\delta_x & {}^{T_i}d_y \\ -{}^{T_i}\delta_y & {}^{T_i}\delta_x & 0 & {}^{T_i}d_z \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\left(\frac{dr}{dt}\right)^2 = \dot{r} \cdot \dot{r} = \text{Trace}\left(\dot{r} \cdot \dot{r}^T\right)$$

$$= \text{Trace}\left\{ \sum_{j=1}^{i} \left(\frac{\partial T_i}{\partial q_j} \cdot \dot{q}_j \cdot {}^i r\right) \cdot \sum_{k=1}^{i} \left(\frac{\partial T_i}{\partial q_k} \cdot \dot{q}_k \cdot {}^i r\right)^T \right\}$$

$$= \text{Trace}\left\{ \sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} \cdot {}^i r \, {}^i r^T \cdot \frac{\partial T_i}{\partial q_k}^T \dot{q}_j \, \dot{q}_k \right\}$$

Kinetic Energy:

$$d_{K_i} = \frac{1}{2} \cdot dm_i \cdot \dot{r} \cdot \dot{r} = \frac{1}{2} \cdot dm_i \cdot \overbrace{\text{trace}\left(\dot{r} \cdot \dot{r}^T\right)}^{tr(\dot{r}\cdot\dot{r}^T)}$$

$$= \frac{1}{2} \cdot dm_i \cdot tr\left(\dot{T}_i \cdot {}^i r \cdot {}^i r^T \cdot \dot{T}_i^T\right)$$

$$= \frac{1}{2} \cdot tr\left(\dot{T}_i \cdot {}^i r \cdot dm_i \cdot {}^i r^T \cdot \dot{T}_i^T\right)$$

$$K_i = \int_{\text{link } i} d_{K_i} = \frac{1}{2} \int tr\left(\dot{T}_i \cdot {}^i r \cdot dm_i \cdot {}^i r^T \cdot \dot{T}_i^T\right)$$

$$= \frac{1}{2} \cdot tr\left[\dot{T}_i \underbrace{\left(\int {}^i r \, {}^i r^T \, dm_i\right)}_{\text{Pseudo Inertia Matrix, } J_i \text{ for link } i.} \cdot \dot{T}_i^T\right]$$

$$= \frac{1}{2} \, tr\left(\dot{T}_i \cdot J_i \cdot \dot{T}_i^T\right)$$

$$K = \sum_{i=1}^{6} K_i = \frac{1}{2} \sum_{i=1}^{6} Tr\left[\sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} \cdot J_i \cdot \frac{\partial T_i}{\partial q_k}^T \cdot \dot{q}_j \cdot \dot{q}_k\right]$$

$$J_i = \int {}^i r \cdot {}^i r^T \cdot dm_i = \begin{bmatrix} \int {}^i x_r^2 \cdot dm_i & \int {}^i x_r \cdot {}^i y_r \cdot dm_i & \int {}^i x_r \cdot {}^i z_r \cdot dm_i & \int {}^i x_r \cdot dm_i \\ \int {}^i x_r \cdot {}^i y_r \cdot dm_i & \int {}^i y_r^2 \cdot dm_i & \int {}^i y_r \cdot {}^i z_r \cdot dm_i & \int {}^i y_r \cdot dm_i \\ \int {}^i x_r \cdot {}^i z_r \cdot dm_i & \int {}^i y_r \cdot {}^i z_r \cdot dm_i & \int {}^i z_r^2 \cdot dm_i & \int {}^i z_r \cdot dm_i \\ \int {}^i x_r \cdot dm_i & \int {}^i y_r \cdot dm_i & \int {}^i z_r \cdot dm_i & \int dm_i \end{bmatrix}$$

$$\left.\begin{aligned} I_{xx} &= \int (y^2+z^2) \cdot dm_i \\ I_{yy} &= \int (x^2+z^2) \cdot dm_i \\ I_{zz} &= \int (x^2+y^2) \cdot dm_i \end{aligned}\right\}$$ second mass moments of inertia of the body about the axes $x, y$ and $z$, respectively.

$$\int x^2 \cdot dm_i = -\frac{1}{2} \cdot \int (y^2+z^2) \cdot dm_i + \frac{1}{2} \int (x^2+z^2) \cdot dm_i + \frac{1}{2} \int (x^2+y^2) \cdot dm_i$$

$$= \frac{1}{2}\left[-I_{xx} + I_{yy} + I_{zz}\right]$$

$$\int y^2 \, dm_i = \frac{1}{2}\left[I_{xx} - I_{yy} + I_{zz}\right]$$

$$\int z^2 \cdot dm_i = \frac{1}{2}\left[I_{xx} + I_{yy} - I_{zz}\right]$$

On the other hand;

$$\left.\begin{array}{l} I_{xy} = \int xy . dm_i \\ I_{xz} = \int x.z . dm_i \\ I_{yz} = \int y.z . dm_i \end{array}\right\}$$ the second mass products (cross products) of inertia of the body.

$$\left.\begin{array}{l} m.\bar{x} = \int x \, dm_i \\ m.\bar{y} = \int y . dm_i \\ m.\bar{z} = \int z . dm_i \end{array}\right\}$$ the first moments of the body.

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$$J_i = \int_{\text{link } i} {}^i r_r \cdot {}^i r_r^T \, dm_i = \begin{bmatrix} \int {}^i x_r^2 \cdot dm_i & \int {}^i x_r \cdot {}^i y_r \, dm_i & \int {}^i x_r \cdot {}^i z_r \cdot dm_i & \int {}^i x_r \cdot dm_i \\ xy & \int {}^i y_r^2 \, dm_i & yz & y \, dm_i \\ xz & yz & \int {}^i z_r^2 \, dm_i & z \, dm_i \\ x \, dm_i & y \, dm_i & z \, dm_i & \int dm_i \end{bmatrix}$$

$$\underbrace{\begin{aligned} I_{xx} &= \int \left({}^i y_r^2 + {}^i z_r^2\right) dm_i \\ I_{yy} &= \int \left({}^i x_r^2 + {}^i z_r^2\right) dm_i \\ I_{zz} &= \int \left({}^i x_r^2 + {}^i y_r^2\right) dm_i \end{aligned}}_{\text{2nd mass moments of the body about the major axes.}} \qquad \underbrace{\begin{aligned} I_{xy} &= \int {}^i x_r \cdot {}^i y_r \cdot dm_i \\ I_{xz} &= \int {}^i x_r \cdot {}^i z_r \cdot dm_i \\ I_{yz} &= \int {}^i y_r \cdot {}^i z_r \cdot dm_i \end{aligned}}_{\text{2nd mass products (cross-products) of inertia of the body.}} \qquad \underbrace{\begin{aligned} m\bar{x} &= \int {}^i x_r \cdot dm_i \\ m\bar{y} &= \int {}^i y_r \cdot dm_i \\ m\bar{z} &= \int {}^i z_r \cdot dm_i \end{aligned}}_{\text{1st moments of the body}}$$

$$J_i = \left[\begin{array}{ccc|c} \frac{1}{2}\left(-{}^i I_{xx} + {}^i I_{yy} + {}^i I_{zz}\right) & {}^i I_{xy} & {}^i I_{xz} & m_i \bar{x}_i \\ {}^i I_{xy} & \frac{1}{2}\left({}^i I_{xx} - {}^i I_{yy} + {}^i I_{zz}\right) & {}^i I_{yz} & m_i \bar{y}_i \\ {}^i I_{xz} & {}^i I_{yz} & \frac{1}{2}\left({}^i I_{xx} + {}^i I_{yy} - {}^i I_{zz}\right) & m_i \bar{z}_i \\ \hline m_i \bar{x}_i & m_i \bar{y}_i & m_i \bar{z}_i & m_i \end{array}\right]$$

symmetric matrix

shows that the link mass $m_i$ is concentrated on the gravity center ${}^i\bar{r}$ is expressed in the $i^{th}$ coord. from

$$K = \frac{1}{2} \sum_{i=1}^{6} \mathrm{Tr} \left\{ \sum_{j=1}^{i} \sum_{k=1}^{i} \frac{\partial T_i}{\partial q_j} \cdot J_i \, \frac{\partial T_i}{\partial q_k}^T \dot{q}_j \, \dot{q}_k \right\}$$

$$K_{act_i} = \frac{1}{2} I_{a_i} \cdot \dot{q}_i^2$$

actuator inertia (If the joint prismatic this then will be equal to the mass)

27.11 (2)

$$
{}^{2}\Delta_1 = \begin{bmatrix} 0 & -{}^{T_2}\delta_{z_1} & {}^{T_2}\delta_{y_1} & {}^{T_2}d_{x_1} \\ {}^{T_2}\delta_{z} & 0 & -{}^{T_2}\delta_{x_1} & {}^{T_2}d_{y_1} \\ -{}^{T_2}\delta_{y_1} & {}^{T_2}\delta_{x_1} & 0 & {}^{T_2}d_{z_1} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & ls_2 \\ 1 & 0 & 0 & lc_2+l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\frac{\partial T_2}{\partial \theta_1} = T_2\,{}^{2}\Delta_1 = \begin{bmatrix} -s_{12} & -c_{12} & 0 & \overset{-(ls_1+ls_{12})}{\cancel{(ls_1+ls_{12})}} \\ c_{12} & -s_{12} & 0 & \cancel{(lc_1+lc_{12})}\ (lc_1+lc_{12}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{yanlış gibi}
$$

Substituting $\frac{\partial T_1}{\partial \theta_1}$ and $\frac{\partial T_2}{\partial \theta_1}$ into $D_{11}$ expression

$$
Tr\left\{\frac{\partial T_1}{\partial \theta_1}\cdot J_1\cdot\frac{\partial T_1^T}{\partial \theta_1}\right\} = Tr\left\{\begin{bmatrix} -s_1 & -c_1 & 0 & -ls_1 \\ c_1 & -s_1 & 0 & lc_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left[\begin{array}{c|c} 0 & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\;0\;0 & m_1 \end{array}\right]\begin{bmatrix} -s_1 & c_1 & 0 & 0 \\ -c_1 & -s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ls_1 & lc_1 & 0 & 0 \end{bmatrix}\right\}
$$

elsi olacak

$$
= m_1\cdot l^2\cdot s_1^2 + m_1\cdot l^2\cdot c_1^2 = m_1\cdot l^2
$$

$$
Tr\left\{\frac{\partial T_2}{\partial \theta_1}\cdot J_2\cdot\frac{\partial T_2^T}{\partial \theta_1}\right\} = Tr\left\{\begin{bmatrix} -s_{12} & -c_{12} & 0 & -(ls_1+ls_{12}) \\ c_{12} & -s_{12} & 0 & lc_1+lc_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left[\begin{array}{c|c} 0 & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\;0\;0 & m_2 \end{array}\right]\begin{bmatrix} -s_{12} & c_{12} & \\ -c_{12} & -s_{12} & \\ 0 & 0 & \\ -(ls_1+ls_{12}) & (lc_1+lc_1 & \end{bmatrix}\right.
$$

Bir yanlışlık var ya da ben yanlışlık yaptım

$$
= Tr\begin{bmatrix} m_2(-ls_1-ls_{12})^2 & & & \\ & m_2(lc_1+lc_{12})^2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}
$$

2 olacak

$$
= m_2\left[l^2+l^2+2l^2\cos(\theta_1+\theta_2-\theta_1)\right] = 2m_2\cdot l^2(1+c_2)
$$

$$
D_{11} = m_1 l^2 + 2m_2\cdot l^2(1+c_2)
$$

Similarly, $D_{22}$ can be obtained.

$$
D_{22} = \sum_{p=\max\{i,j\}}^{2} Tr\left\{\frac{\partial T_p}{\partial q_j}\cdot J_p\cdot\frac{\partial T_p^T}{\partial q_j}\right\} = Tr\left\{\frac{\partial\,{}^1T_2}{\partial \theta_2}\cdot J_2\cdot\frac{\partial\,{}^1T_2^T}{\partial \theta_2}\right\}
$$

both 2

$$
\frac{\partial\,{}^1T_2}{\partial \theta_2} = {}^1T_2\;{}^2\Delta_2
$$

$$
{}^1d_2 = 0\,,\quad {}^1\delta_2 = 1.k
$$

$$
\begin{array}{c} \\ A_2 = \\ \uparrow \\ {}^1T_2 \end{array}
\begin{array}{c} \begin{matrix} {}^1n_2 & {}^1o_2 & {}^1a_2 & {}^1p_2 \end{matrix} \\ \begin{bmatrix} c_2 & -s_2 & 0 & lc_2 \\ s_2 & c_2 & 0 & ls_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \phi \end{bmatrix} \end{array}
$$

doğrusu bu

## POTENTIAL ENERGY:

The potential energy of an object with mass m at a height h above some zero reference elevation in a gravity field $\bar{g}$ is

$$P = -m.\bar{g}.h$$

Let the center of gravity of the object be located at $\bar{r}$ then

$$P = -m.\bar{g}.\bar{r}$$

$\bar{r} = 10i + 20j + 30k$

$\bar{g} = 0i + 0j + 9.8k$ Sniye (-) değil

$P = 294$ (N-m)

The potential energy $p_i$ of link i, whose center of mass $m_i$ is described by $\bar{r}_i$ w.r.t link i coordinate frame, $T_i$ is then

$$P_i = -m_i . \bar{g}^T . T_i . {}^{T}\bar{r}_i$$

$$\bar{g} = [g_x \; g_y \; g_z \; 0]^T$$

$$P = \sum_{i=1}^{6} -m_i . \bar{g}_i^T . T_i . {}^{i}\bar{r}_i$$ → Total Potential Energy

Lagrangian:

$$L = K - P = \sum_{i=1}^{n} k_i - \sum_{i=1}^{n} P_i$$

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}$$

After necessary substitution and simplifications, we have

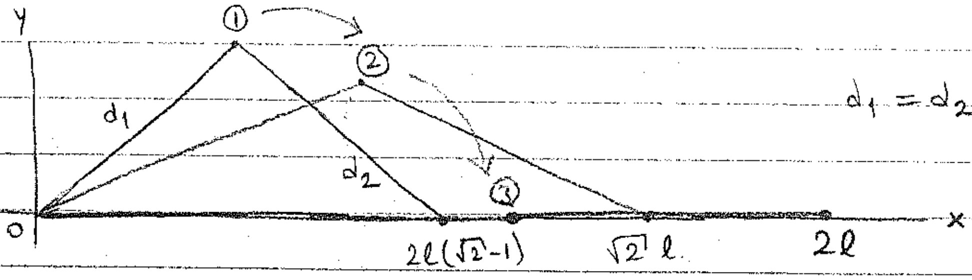
$$F_i = \overbrace{\sum_{j=1}^{n} D_{ij}.\ddot{q}_j}^{\text{coupling term}} + \overbrace{I_{a_i}.\ddot{q}_i}^{\text{actuator inertia term}} + \underbrace{\sum_{j=1}^{n}\sum_{k=1}^{n} D_{ijk}.\dot{q}_j\,\dot{q}_k}_{\text{corriolis coeff}} + \underbrace{D_i}_{\text{gravity term}}$$

where $$D_{ij} = \sum_{p=\max(i,j)}^{n} Tr\left(\frac{\partial T_p}{\partial q_j} J_p \frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_{ijk} = \sum_{p=\max(i,j,k)}^{n} Tr\left(\frac{\partial^2 T_p}{\partial q_j \partial q_k} . J_p . \frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_i = \sum_{p=i}^{n} -m_p \bar{g}^T \frac{\partial T_p}{\partial q_i} {}^{p}\bar{r}_p$$

Example:

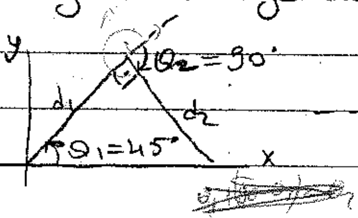



$d_1 = d_2 = \ell$

Consider the above two link manipulator whose link lengths and masses are equal. Assume that the arm manipulator is first in position 1 with the tip being at $x = 2\ell(\sqrt{2}-1)$ point on the x-axis. It's then moved to position 2 with the tip this time being at $\sqrt{2}\,\ell$ where the actuator switches from acceleration to deceleration. It finally holds a position 3 where the tip is at $x = 2\ell$. Find the joint angular velocities in terms of linear velocities along x and y-axes.

Soln:

$$x = \ell.\cos\theta_1 + \ell\cos(\theta_1+\theta_2)$$
$$= \ell.\cos 45 + \ell\cos(45-90) = \sqrt{2}\,\ell$$




--- 2

$$x = \ell.\cos 0 + \ell.\cos 0 = 2\ell \quad \text{--- } 3$$
$$\Rightarrow \theta_1 = 0, \ \theta_2 = 0$$

$$x = \ell.\cos\theta_1 + \ell.\cos(\theta_1+\theta_2) = 2\ell.\cos\theta_1 = 2\ell(\sqrt{2}-1) \quad \text{--- } 1$$
$$\Rightarrow \theta_1 = \cos^{-1}(\sqrt{2}-1)$$

$$y = \ell.\sin\theta_1 + \ell.\sin(\theta_1+\theta_2)$$

$$\dot{x} = -\ell.\sin\theta_1.\dot{\theta}_1 - \ell.\sin(\theta_1+\theta_2).(\dot{\theta}_1+\dot{\theta}_2)$$
$$= -\ell.(\sin\theta_1 + \sin(\theta_1+\theta_2)).\dot{\theta}_1 - \ell.\sin(\theta_1+\theta_2).\dot{\theta}_2$$

$$\dot{y} = \ell.\cos\theta_1.\dot{\theta}_1 + \ell.\cos(\theta_1+\theta_2).(\dot{\theta}_1+\dot{\theta}_2)$$
$$= \ell.(\cos\theta_1 + \cos(\theta_1+\theta_2)).\dot{\theta}_1 + \ell.\cos(\theta_1+\theta_2).\dot{\theta}_2$$

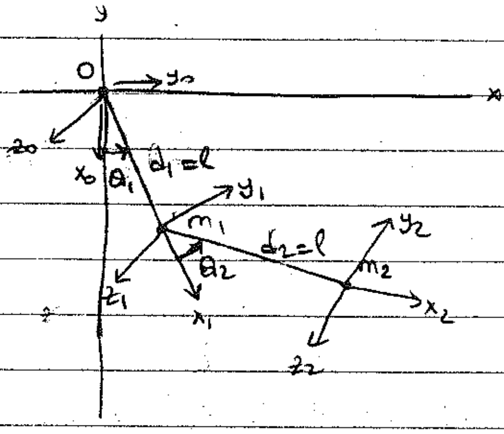
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix} -\ell(\sin\theta_1 + \sin(\theta_1+\theta_2)) & -\ell.\sin(\theta_1+\theta_2) \\ \ell.(\cos\theta_1 + \cos(\theta_1+\theta_2)) & \ell.\cos(\theta_1+\theta_2)\end{bmatrix}.\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \frac{1}{\ell^2.\sin\theta_2}.\begin{bmatrix} \ell\cos(\theta_1+\theta_2) & \ell\sin(\theta_1+\theta_2) \\ -\ell.(\cos\theta_1 + \cos(\theta_1+\theta_2)) & -\ell(\sin\theta_1+\sin(\theta_1+\theta_2))\end{bmatrix}.\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}$$

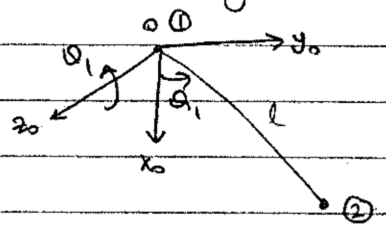
(doğru)

Example:



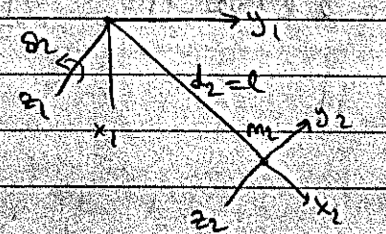

Compute the effective inertia and coupling inertia and the gravity term for the two link manipulator described in one of the previous examples.

Link 1: Joint axes are //. Choose joint 2 axis as origin; $z_1$ so joint 2 axis; $x_1$ is along the perpendicular (common normal) between $z_0$ and $z_1$. Then the position of joint 2 w.r.t $(x_0, y_0, z_0)$ coordinate system is given then as:




$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & l.\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l.\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Link 2: Joint axes 2 and 3 are //. Choose joint ③ as origin and therefore $x_2$ is along $d_2$ since it is the common normal between joint axes 2 and 3. Hence




$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & lc_2 \\ s_2 & c_2 & 0 & ls_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_2 = A_1 A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & (lc_1 + lc_{12}) \\ s_{12} & c_{12} & 0 & (ls_1 + ls_{12}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D_{11} = \sum_{p=1}^{2} Tr\left(\frac{\partial T_p}{\partial q_1} . J_p . \frac{\partial T_p^T}{\partial q_1}\right) \qquad q_1 = \theta_1 \ \& \ q_2 = \theta_2$$

$$J_1 = \int {}^1r . {}^1r . dm_1 = \begin{bmatrix} \int {}^1x^2 . dm & \int {}^1x . {}^1y . dm & \int {}^1x . {}^1z . dm & \int {}^1x . dm \\ \int {}^1x . {}^1y . dm & \int {}^1y^2 . dm & \int {}^1y . {}^1z . dm & \int {}^1y . dm \\ \int {}^1x . {}^1z . dm & \int {}^1y . {}^1z . dm & \int {}^1z^2 . dm & \int {}^1z . dm \\ \int {}^1x . dm & \int {}^1y . dm & \int {}^1z . dm & \int dm \end{bmatrix}$$

$${}^1x = {}^1y = {}^1z = 0$$

$$\Rightarrow J_1 = \left[\begin{array}{ccc|c} & 0 & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & m_1 \end{array}\right]$$

Due to the same reasons of $J_1$; $J_2$ is equal to:

$$J_2 = \left[\begin{array}{ccc|c} & 0 & & 0 \\ & & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & m_2 \end{array}\right]$$

$$\frac{\partial T_1}{\partial \theta_1} = \underset{A_1}{{}^0T_1} \cdot {}^{0}T_1\Delta_1 \qquad d_1 = 0i + 0j + 0k$$
$$\delta_1 = 0i + 0j + 1k$$

revolute joint değil mi acaba
evet o yüzden
niye bu şekilde

$$\Rightarrow A_1 = \begin{array}{c} \begin{array}{cccc} n_1 & o_1 & a_1 & p_1 \end{array} \\ \begin{bmatrix} c_1 & -s_1 & 0 & lc_1 \\ s_1 & c_1 & 0 & ls_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\delta_1 \times p_1 = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ lc_1 & ls_1 & 0 \end{vmatrix} = -l s_1 i + l c_1 j$$

$$\left[(\delta_1 \times p_1) + d_1\right] = -l s_1 . i + l . c_1 j$$

$${}^1d_{x_1} = n_i . \left[(\delta_1 \times p_1) + d_1\right] = \begin{bmatrix} c_1 & s_1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} -l.s_1 \\ l.c_1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$${}^1d_{y_1} = o_i . \left[(\delta_1 \times p_1) + d_1\right] = \begin{bmatrix} -s_1 & c_1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} -ls_1 \\ lc_1 \\ 0 \\ 0 \end{bmatrix} = l$$

$${}^1d_{z_1} = a_i . \left[(\delta_1 \times p_1) + d_1\right] = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} . \begin{bmatrix} -ls_1 \\ lc_1 \\ 0 \\ 0 \end{bmatrix} = 0$$

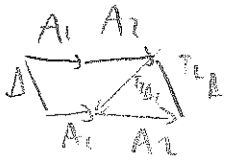
$${}^1\delta_{x_1} = n_1 . \delta_1 = 0$$
$${}^1\delta_{y_1} = o_1 . \delta_1 = 0$$
$${}^1\delta_{z_1} = a_1 . \delta_1 = 1$$

$${}^{0}T_1\Delta_1 = \begin{bmatrix} 0 & -{}^1\delta_{z_1} & {}^1\delta_{y_1} & {}^1d_{x_1} \\ {}^1\delta_{z_1} & 0 & -{}^1\delta_{x_1} & {}^1d_{y_1} \\ -{}^1\delta_{y_1} & {}^1\delta_{x_1} & 0 & {}^1d_{z_1} \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\, {}^0T_1}{\partial \theta_1} = {}^0T_1 \cdot {}^{{}^0T_1}\Delta_1 = \begin{bmatrix} -s_1 & -c_1 & 0 & -ls_1 \\ c_1 & -s_1 & 0 & lc_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & -s_1 & 0 & lc_1 \\ s_1 & c_1 & 0 & ls_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute secondly;

$$\frac{\partial\, {}^0T_2}{\partial \theta_1} = {}^0T_2 \cdot {}^{T_2}\Delta_1$$



$$\Delta A_1 A_2 = A_1 A_2 \, {}^{T_2}\Delta$$

$$\Delta A_1 = A_1 A_2 \, {}^{T_2}\Delta_1$$

$${}^{T_2}\Delta_1 = A^{-1} \Delta A_1$$

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$\delta_2 = 0i + 0j + 1k$

$d_1 = 0i + 0j + 0k$

$$T_2 = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} n_2 & o_2 & a_2 & p_2 \end{array}$$

$$T_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & lc_1 + lc_{12} \\ s_{12} & c_{12} & 0 & ls_1 + ls_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\delta_2 \times p_2 = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ lc_1 + lc_{12} & ls_1 + ls_{12} & 0 \end{vmatrix} = -(ls_1 + ls_{12})i + (lc_1 + lc_{12})j$$

snıye dı ellenıyo' (devam)

$$[(\delta_2 \times p_2) + d_1] = -(ls_1 + ls_{12})i + (lc_1 + lc_{12})$$

$${}^2d_{x_1} = n_2 * [\quad] = [c_{12} \;\; s_{12} \;\; 0 \;\; 0] \begin{bmatrix} -(ls_1 + ls_{12}) \\ (lc_1 + lc_{12}) \\ 0 \\ 0 \end{bmatrix}$$

$${}^2d_{x_1} = lc_1 s_{12} - ls_1 c_{12} =$$

$${}^2d_{x_1} = l.\sin(\theta_1 + \theta_2 - \theta_1) = ls_2$$

$${}^2d_{y_1} = o_2 * [\;\cdot\;] = [-s_{12} \;\; c_{12} \;\; 0 \;\; 0] \begin{bmatrix} -(l_1+l_2) \\ (l_1+l_2) \\ 0 \\ 0 \end{bmatrix} = ls_1 . s_{12} + lc_1 c_{12} = lc_2$$

$$+ l$$

$${}^2d_{z_1} = a_2 * [\cdot] = [0 \;\; 0 \;\; 1 \;\; 0] \begin{bmatrix} -(\;) \\ (\;) \\ 0 \\ 0 \end{bmatrix} = 0$$

$${}^2\delta_{x_1} = n_2 . \delta_2 = [c_{12} \;\; s_{12} \;\; 0] . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$${}^2\delta_{y_1} = o_2 . \delta_2 = [-s_{12} \;\; c_{12} \;\; 0] . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$${}^{T_2}\Delta_1 = \begin{bmatrix} 0 & -1 & 0 & ls_2 \\ 1 & 0 & 0 & lc_2 + l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$${}^2\delta_{z_1} = a_2 . \delta_2 = [0 \;\; 0 \;\; 1] . \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1$$

$$\begin{bmatrix} -s_{12} & -c_{12} & 0 & ls_1 \\ c_{12} & -s_{12} & 0 & lc_1 \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{bmatrix} = T_2 \, {}^2\Delta_1 = \begin{pmatrix} c_{12} & -s_{12} & 0 & lc_1 + lc_{12} \\ s_{12} & c_{12} & 0 & ls_1 + ls_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & ls_2 \\ 1 & 0 & 0 & lc_2 + l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} -s_{12} & -c_{12} & 0 & lc_{12}s_2 - lc_2 s_{12} \rightarrow ls_1 \\ c_{12} & -s_{12} & 0 & ls_{12}s_2 + lc_{12}c_2 + lc_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
{}^{1}\delta_2 \times {}^{1}p_2 = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ lc_2 & ls_2 & 0 \end{vmatrix} = -ls_2\, i + lc_2 . k
$$

$$
\left[ ({}^{1}\delta_2 \times {}^{1}p_2) + {}^{1}d_2 \right] = -ls_2 . i + lc_2 . j
$$

$$
{}^{2}d_{x_2} = {}^{1}n_2 . [\ldots] = [c_2 \;\; s_2 \;\; 0] \begin{bmatrix} -ls_2 \\ -lc_2 \\ 0 \end{bmatrix} = 0
$$

$$
{}^{2}d_{y_2} = {}^{1}o_2 . [\ldots] = [-s_2 \;\; c_2 \;\; 0] . \begin{bmatrix} \downarrow \end{bmatrix} = l
$$

$$
{}^{2}d_{z_2} = {}^{1}a_2 . [\ldots] = [0 \;\; 0 \;\; 1] . \begin{bmatrix} \;\; \end{bmatrix} = 0
$$

$$
{}^{2}\delta_{x_2} = {}^{1}n_2 . \delta_2 = 0
$$

$$
{}^{2}\delta_{y_2} = {}^{1}o_2 . \delta_2 = 0
$$

$$
{}^{2}\delta_{z_2} = {}^{1}a_2 . \delta_2 = 1
$$

$$
{}^{2}\Delta_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

yanlış

$$
\begin{pmatrix} -s_2 & -c_2 & 0 & -ls_2 \\ c_2 & -s_2 & 0 & lc_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\frac{\partial T_2}{\partial \theta_2} = {}^{1}T_2 . {}^{2}\Delta_2 = \begin{bmatrix} -s_2 & -c_2 & 0 & -ls_2 \\ c_2 & -s_2 & 0 & lc_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_2 & -s_2 & 0 & lc_2 \\ s_2 & c_2 & 0 & ls_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
D_{22} = Tr \left\{ \begin{bmatrix} -s_2 & -c_2 & 0 & -ls_2 \\ c_2 & -s_2 & 0 & lc_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\,0\,0 & m_2 \end{bmatrix} . \begin{bmatrix} -s_2 & c_2 & 0 & 0 \\ -c_2 & -s_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ls_2 & lc_2 & 0 & 0 \end{bmatrix} \right\}
$$

$$
= Tr \left\{ \begin{bmatrix} m_2 l^2 s_2^2 & & & \\ & m_2 l^2 c_2^2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \right\} = m_2 . l^2
$$

ama sonuç değişmiyor

$$
D_{12} = Tr \left\{ \frac{\overset{0}{\partial} T_2}{\partial \theta_2} . J_2 . \frac{\overset{0}{\partial} T_2^{T}}{\partial \theta_1} \right\} = Tr \left\{ \begin{bmatrix} -s_2 & \overset{-\text{olacak}}{c_2} & 0 & \overset{\text{değişir}}{-ls_2} \\ c_2 & -s_2 & 0 & lc_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} -s_{12} & c_{12} & 0 & 0 \\ -c_{12} & -s_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -(ls_1 + ls_{12}) & (lc_1 + lc_{12}) & 0 & 0 \end{bmatrix} \right\}
$$

bulunmalı (yanlış olabilir) $-ls_1$ $lc_1$