$$D_{12} = Tr\left\{\begin{bmatrix} m_2 l s_2 (l s_1 + l s_{12}) & & & \\ & m_2 l c_2 (l c_1 + l c_{12}) & & \\ & & 0 & \\ & & & 0 \end{bmatrix}\right\}$$

$$\Rightarrow D_{12} = m_2 l^2 \left[\cos(\theta_1 - \theta_2) + c_1\right]$$

02.12.2002 / Monday

ANGULAR VELOCITY AND ACCELERATION:

"Robot Dynamics and Control", M.W. Spong, M. Vidyasagar, Wiley pp 50-56

Defining skew symmetric matrix:

$$S + S^T = 0$$

$$s_{ij} + s_{ji} = 0 \qquad i, j = 1, 2, 3, \ldots$$

$$s_{ii} = 0$$

$$S(\underline{s}) = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix} \quad \text{for } \underline{s} = s_1 i + s_2 j + s_3 k$$

$$S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{for } i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$S(j) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad \text{for } j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$S(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for } k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Properties of Skew Symmetric Matrices:

i. The linearity property

For any two vectors $\underline{a}$ and $\underline{b} \in \mathbb{R}^3$ and scalars α and β

$$S(\alpha \underline{a} + \beta \underline{b}) = \alpha S(\underline{a}) + \beta . S(\underline{b})$$

ii. For a $\underline{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$ and $\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$

$$S(\underline{a})\,\underline{p} = \underline{a} \times \underline{p}$$

iii. If $\underline{a}, \underline{b} \in \mathbb{R}^3$ then

$$R(\underline{a} \times \underline{b}) = R\underline{a} \times R\underline{b}$$

If R is an orthogonal rotation matrix.

iv. For two vectors $\underline{a}$ and $\underline{b}$

$$R\,S(\underline{a})\,R^T \underline{b} = R\left(\underline{a} \times R^T \underline{b}\right) \quad \text{--- (ii)}$$

$$= (R\underline{a}) \times (\underbrace{RR^T}_{I}\underline{b}) \quad \text{--- (ii)}$$

$$= (R\underline{a}) \times \underline{b}$$

$$= S(R\underline{a})\,\underline{b}$$

Therefore, $R\,S(\underline{a})\,R^T = S(R\underline{a})$ ---> similarity transformation of $S(\underline{a})$

For example, $R = Rot(\theta)$

$$Rot(\theta) \cdot Rot^T(\theta) = I$$

$$\underbrace{\frac{dR}{d\theta} \cdot R^T(\theta)}_{S^T} + \underbrace{R(\theta) \cdot \frac{dR^T(\theta)}{d\theta}}_{S^T} = 0$$

$$S \triangleq \frac{dR}{d\theta} \cdot R^T(\theta)$$

$$S^T \triangleq \left(\frac{dR}{d\theta} R^T(\theta)\right)^T = R(\theta) \cdot \frac{dR^T(\theta)}{d\theta}$$

$$\frac{dR}{d\theta} \cdot \underbrace{R^T(\theta) \cdot R(\theta)}_{I} = S\,R(\theta)$$

$$\frac{dR}{d\theta} = S\,R(\theta)$$

Example: Let $R = R(\theta) = Rot(x, \theta)$

$$S = \frac{dR}{d\theta} R^T(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i)$$

Thus;
$$\frac{d\,Rot(x,\theta)}{d\theta} = S(i) \,.\, Rot(x,\theta)$$

$$\frac{d\,Rot(y,\theta)}{d\theta} = S(j) \,.\, Rot(y,\theta)$$

$$\frac{d\,Rot(z,\theta)}{d\theta} = S(k) \,.\, Rot(z,\theta)$$

Suppose that a rotation matrix R is time varying so that $R = R(t)$ for every $t \in R$.

$$\mathring{R}(t) = S(t) \,.\, R(t)$$

Example: Suppose that $R(t) = Rot(x, \theta(t))$

$$\dot{R}(t) = \frac{dR}{dt} = \frac{dR}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta} \,.\, S(i) \,.\, R(t) = S(w(t)) \,.\, R(t)$$

where $w(t) = i \,.\, \dot{\theta}$ angular velocity about the x-axis.

Suppose P_1 is a vector fixed in a coordinate system $(o_1\ x_1\ y_1\ z_1)$ and the frame is rotating relative to the frame $(o\ x_0\ y_0\ z_0)$

$$P_0 = R(t) \,.\, P_1$$

velocity of P_0 is $\mathring{P}_0$:

$$\mathring{P}_0 = S(\underline{w}) \,.\, R(t) \,.\, P_1$$

$$\mathring{P}_0 = S(\underline{w}) \,.\, P_0 = \underline{w} \times P_0$$

Suppose that the homogenous transformation relating the above two frames is time dependent so that

$$H(t) = \begin{bmatrix} \overbrace{R(t)}^{\text{Rotationary part}} & \overbrace{d(t)}^{\text{translationary part}} \\ \underbrace{0}_{\substack{\text{zero row} \\ 1\times 3}} & 1 \end{bmatrix}$$

$P_0 = R(t) \cdot P_1 + d(t)$

$\dot{P}_0 = \dot{R}(t) \cdot P_1 + \dot{d}(t) = S(\underline{\omega}) \cdot R(t) \cdot P_1 + \dot{d}(t) = \underline{\omega} \times (RP_1) + \dot{d}(t)$

$r \triangleq RP_1$ the vector from O_1 to P_0 expressed in $(O\ x_0\ y_0\ z_0)$, the orientation of frame.

nokta olacak mı

$v \triangleq \dot{d}(t) + \underbrace{R(t) \cdot P_1}$ the rate of which the origin O_1 is moving.

it is the rate of change of P_1, expressed in the frame $(O\ x_0\ y_0\ z_0)$

If the vector P_1 is also changing relative to the frame $(O_1\ x_1\ y_1\ z_1)$ the term
nokta olacak mı acaba
$R(t)\dot{P}_1$ is to be added to the term V.

then $V \triangleq \dot{d}(t) + \dot{R}(t)P_1 + R(t)\dot{P}_1$

For the acceleration, note that:

$$\frac{d(\underline{a} \times \underline{b})}{dt} = \frac{d\underline{a}}{dt} \times b + \underline{a} \times \frac{d\underline{b}}{dt}$$

$$\dot{P}_0 = \dot{R}P_1 + \dot{d} = (\underline{\omega} \times RP_1) + \dot{d}$$

$$\ddot{P}_0 = \dot{\omega} \times RP_1 + \omega \times (\dot{R}P_1) + \ddot{d}$$

$$= \underbrace{\dot{\omega} \times r}_{\text{Transverse acceleration}} + \underbrace{\omega \times (\omega \times r)}_{\text{Centripetal acceleration}} + \underset{\text{linear acceleration}}{a}$$

Centripetal acceleration of the particle and it is always directed toward the axis of rotation and is $\perp$ to the axis.

If P_1 is changing w.r.to $(O_1\ x_1\ y_1\ z_1)$ then the last eqn becomes;

$$\ddot{P}_0 = \dot{\underline{\omega}} \times r + \underline{\omega} \times (\underline{\omega} \times r) + \underbrace{2\underline{\omega} \times R\dot{P}_1}_{\text{Coriolis acceleration}} + a$$

$a = R \cdot \ddot{P}_1 + \ddot{d}$

Addition of Angular Velocities:

$$P_0 = {}^0R_1 \cdot P_1 + {}^0d_1$$

$$P_1 = {}^1R_2 \cdot P_2 + {}^1d_2$$

$$P_0 = {}^0R_2 \cdot P_2 + {}^0d_2$$

$${}^0R_2 = {}^0R_1 \cdot {}^1R_2$$

$${}^0d_2 = {}^0d_1 + {}^0R_1 \cdot {}^1d_2$$

$${}^0\dot{R}_2 = {}^0\dot{R}_1 \cdot {}^1R_2 + {}^0R_1 \cdot {}^1\dot{R}_2$$

$${}^0\dot{R}_2 = S({}^0w_2) \cdot {}^0R_2$$

$${}^0\dot{R}_1 \cdot {}^1R_2 = S({}^0w_1) \cdot {}^0R_1 \cdot {}^1R_2 = S(w_1) \cdot {}^0R_2$$

$$\begin{aligned} {}^0R_1 \cdot {}^1\dot{R}_2 &= {}^0R_1 \cdot S({}^1w_2) \cdot {}^1R_2 = {}^0R_1 \, S({}^1w_2) \cdot {}^0R_1^T \cdot {}^0R_1 \cdot {}^1R_2 \\ &= S({}^0R_1 \cdot {}^1w_2)\, {}^0R_1 \cdot {}^1R_2 = S({}^0R_1 \cdot {}^1w_2)\, {}^0R_2 \end{aligned}$$

$$S({}^0w_2) \cdot {}^0R_2 = \left[S({}^0w_1) + S({}^0R_1 \, {}^1w_2) \right] \cdot {}^0R_2$$

$$S(\underline{a}) + S(\underline{b}) = S(\underline{a} + \underline{b})$$

$$S({}^0w_2) \cdot {}^0R_2 = S({}^0w_1 + {}^0R_1 \cdot {}^1w_2) \cdot {}^0R_2$$

$${}^0w_2 = {}^0w_1 + {}^0R_1 \cdot {}^1w_2$$

Generalization to n many frames,

$${}^0R_n = {}^0R_1 \cdot {}^1R_2 \cdots {}^{n-1}R_n$$

$${}^0\dot{R}_n = S({}^0w_n) \cdot {}^0R_n$$

$$\boxed{{}^0w_n = {}^0w_1 + {}^0R_1 \, {}^1w_2 + {}^0R_2 \cdot {}^2w_3 + \cdots + {}^0R_{n-1} \cdot {}^{n-1}w_n}$$

Inverse velocity and acceleration:

Consider an n-link manipulator with joint variables $q_1 \cdots q_n$. Let

$${}^0T_n(\underline{q}) = \left[\begin{array}{c|c} {}^0R_n(\underline{q}) & {}^0d_n(\underline{q}) \\ \hline \underline{0} & 1 \end{array} \right]$$

$$S({}^0w_n) = {}^0\dot{R}_n \cdot ({}^0R_n)^T$$

$${}^0v_n = {}^0\dot{d}_n$$

$${}^0v_n = J_v \cdot \underline{\dot{q}} \qquad \Rightarrow \qquad {}^0J_n = \left[\begin{array}{c} J_v \\ \hline J_w \end{array} \right] \quad \text{Manipulator Jacobian}$$

$${}^0w_n = J_w \cdot \underline{\dot{q}}$$

$$\left[\begin{array}{c} {}^0v_n \\ \hline {}^0w_n \end{array} \right] = {}^0J_n \cdot \underline{\dot{q}}$$

$$\Downarrow$$

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \\ \delta x/dt \\ \delta y/dt \\ \delta z/dt \end{bmatrix} = \begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \cdot \begin{bmatrix} dq_1/dt \\ dq_2/dt \\ \vdots \\ dq_n/dt \end{bmatrix}$$

Recall that angular velocities can be added vectorially provided that they're expressed relative to a common coordinate frame. Thus the angular velocity of end-effector relative to the base by expressing the angular velocity of the each link in the orientation of the base frame and then summing them.

If the i^{th} joint is revolute then the ith joint variable q_i equals θ_i and the axis of rotation is z_{i-1} axis. Thus the angular velocity of link i expressed in the frame $i-1$ is given by

$$^{i-1}\omega_i = \underset{\dot{\theta}_i}{\underset{\downarrow}{\dot{q}_i}} \cdot k \leftarrow \text{unit vector in frame } i-1$$

If the i^{th} joint is prismatic then the motion of frame i relative to frame $i-1$ is a translation and $^{i-1}\omega_i = 0$. Since $q_i = d_i$, now there is no rotation.

—o—

The overall angular velocity of the end-effector $^0\omega_n$ in the base frame is then

$$^0\omega_n = \rho_1 \cdot \dot{q}_1 \cdot k + \rho_2 \cdot \dot{q}_2 \cdot {}^0R_1 \cdot k + \cdots + \rho_n \cdot \dot{q}_n \cdot R_{n-1} \cdot k$$

$$^0\omega_n = \sum_{i=1}^{n} \rho_i \cdot \dot{q}_i \cdot z_{i-1}$$

where $z_{i-1} = {}^0R_{i-1} \cdot k$ and $\rho_i = 1$ if the joint is revolute; and equal to 0 if the joint is prismatic.

Note that, $z_0 = k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The lower part of Jacobian, J_ω, is thus given by

$$J_\omega = \left[\rho_1 z_0 \quad \rho_2 z_1 \ \cdots \ \rho_n \cdot z_{n-1} \right]$$

The linear velocity of the end-effector is just $^0\dot{d}_n$. By the chain rule for differentiation

$$^0\dot{d}_n = \sum_{i=1}^{n} \frac{\partial d_n}{\partial q_i} \cdot \dot{q}_i$$

Thus the i^{th} column of J_v is just $\frac{\partial \dot{d}_n}{\partial q_i}$. furthermore, this expression is just the linear velocity of the end-effector that would result if $\dot{q}_i$ is equal to one and the other $\dot{q}_j$ are zero. In other words, the i^{th} column of the Jacobian is generated by holding all joints fixed but the i^{th} and actuating the i^{th} at the unit velocity

Inverse velocity and acceleration:

$$^{0}v_n = J_v \dot{q} \qquad {}^{0}J_n = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$$
$$^{0}w_n = J_w \dot{q}$$

$$^{0}w_n = \rho_1 \dot{q}_1 k + \rho_2 \dot{q}_2 \, {}^{0}R_1 + \dots + \rho_n \dot{q}_n \, {}^{0}R_{n-1} k = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}$$

$$z_{i-1} = {}^{0}R_{i-1} k$$

$\rho_i = 1$ if the joint is revolute.
$= 0$ if the joint is prismatic.

$$J_w = \left[\rho_1 z_0 \quad \rho_2 z_1 \quad \dots \quad \rho_n z_{n-1} \right]$$

$$^{0}\dot{d}_n = \sum_{i=1}^{n} \frac{\partial\, {}^{0}d_n}{\partial q_i} \dot{q}_i$$

$\rho_n \dot{q}_n \, {}^{0}R_{n-1} k$

J is just $\dfrac{\partial \dot{d}_n}{\partial a_i}$.

$^{0}\dot{d}_n$ is just the linear velocity of the end-effector that would result if $\dot{q}_i$ is equal to "1" and the other $\dot{q}_j$ are "0".

Thus the i-th column of $^{0}J_n$ is generated by holding all joints fixed but the i-th, and actuating the i-th at unit velocity. Consider these two cases separetely.

Case 1

If joint i prismatic, then $^{0}R_{j-1}$ is independent of $q_i = d_i$ for all j and

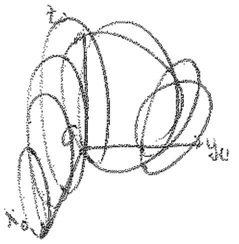
$$\boxed{^{i-1}d_i = d_i k + {}^{i-1}R_i \, \partial_i \, i}$$

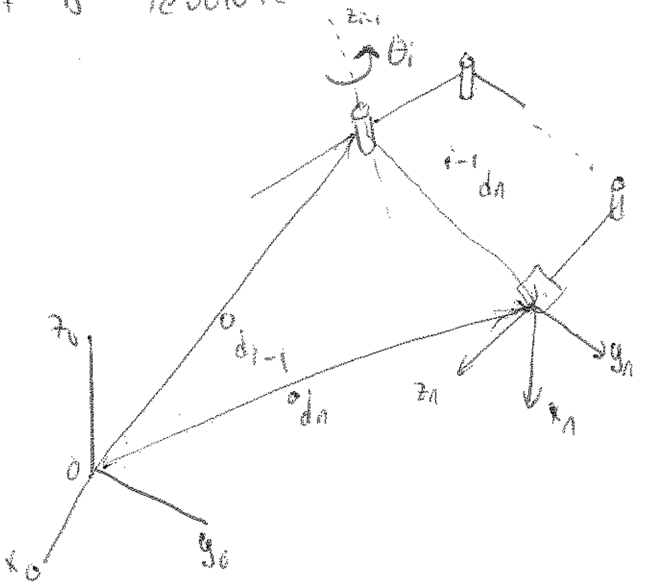
If all joints are fixed but the i-th we have that

$$^{0}\dot{d}_n = {}^{0}R_{i-1} \, {}^{i-1}\dot{d}_i = {}^{0}\dot{d}_i \, {}^{0}R_{i-1} k = {}^{0}\dot{d}_i \, z_{i-1} \longrightarrow \frac{\partial\, {}^{0}d_n}{\partial q_i} = z_{i-1}$$

Case 2

If the joint is revolute





Vector ^0d_k from the origin O_0 to the origin O_k for any k and write

$$* \quad {}^0d_n = {}^0d_{i-1} + {}^0R_{i-1}\,{}^{i-1}d_n$$

or in the new notation

$$* \quad o_n - o_{i-1} = {}^0R_{i-1}\,{}^{i-1}d_n$$

Refering to the scheme we consider, note that both $^0d_{i-1}$ and $^0R_{i-1}$ are constant if only the i-th joint is actuated.

Therefore

$$^0\dot{d}_n = {}^0R_{i-1}\,{}^{i-1}\dot{d}_n$$

Since the motion of link i is a rotation q_i about z_{i-1} we have

$$^{i-1}\dot{d}_n = \dot{q}_i k \times {}^{i-1}d_n$$

and thus

$$^0\dot{d}_n = {}^0R_{i-1}\left(\dot{q}k \times {}^{i-1}d_n\right) = \dot{q}_i\,{}^0R_{i-1}k \times {}^0R_{i-1}\,{}^{i-1}d_n = \dot{q}_i\, z_{i-1} \times (o_n - o_{i-1})$$

Hence

$$\frac{\partial\, {}^0d_n}{\partial q_i} = z_{i-1} \times (o_n - o_{i-1})$$

$$J_v = \begin{bmatrix} J_{v_1} & \cdots & J_{v_n} \end{bmatrix}$$

$J_{v_i} = z_{i-1} \times (o_n - o_{i-1}) \longrightarrow$ if joint is revolute

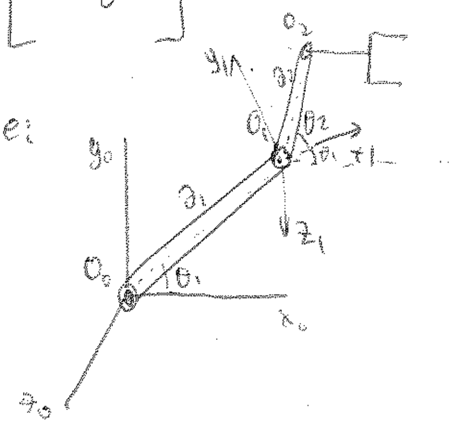
$J_{v_i} = z_{i-1} \longrightarrow$ prismatic

$$J = \begin{bmatrix} J_1, J_2, \cdots & J_i \end{bmatrix}$$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_n - o_{i-1}) \\ z_{i-1} \end{bmatrix} \text{ for revolute}$$

$$= \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} \text{ for prismatic}$$

Example:



Both joints are revolute.
The Jacobian in this case is of the form

$$\begin{array}{ccc} & \text{Joint 1 } i=1 & \text{Joint 2 } i=2 \\ J(q) = & \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix} \end{array}$$

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad O_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$z_0 = 0_i + 0_j + 1k$

$z_1 = 0i + 0j + 1k$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1 c_1 & a_2 c_1 & 0 \end{vmatrix}$$

$o_2 - o_0 = o_2 \qquad o_2 - o_1 = \begin{bmatrix} a_2 c_{12} \\ a_2 s_{12} \\ 0 \end{bmatrix}$

$$J = \begin{bmatrix} (-a_1 s_1 - a_2 s_{12}) & -a_2 s_{12} \\ (a_1 c_1 + a_2 c_{12}) & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$[dx] = J[dq]$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2 c_{12} & a_2 s_{12} & 0 \end{vmatrix} = -i a_2 s_{12} - j$$

587 (6) 9-12-2002

$$\begin{bmatrix} {}^T d \\ {}^T \delta \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & (p\times n)_y & (p\times n)_z \\ o_x & o_y & o_z & (p\times o)_x & (p\times o)_y & (p\times o)_z \\ a_x & a_y & a_z & (p\times a)_x & (p\times a)_y & (p\times a)_z \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & o_x & o_y & o_z \\ 0 & 0 & 0 & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} {}^0 d \\ \\ \\ {}^0 \delta \\ \\ \\ \end{bmatrix}$$

$n \cdot ((\delta \times p) + d)$

Example: STANFORD MANIPULATOR

Joint 3 is prismatic $o_3 = o_4 = o_5 = 0$

as a consequence of the spherical wrist and the coordinate frame assignment.

$$J = \left[\begin{array}{c|c|c|c|c|c} z_0 \times (o_6 - o_0) & z_1 \times (o_6 - o_1) & z_2 & z_3 \times (o_6 - o) & z_4 \times (o_6 - o) & z_5 \times (o_6 - o) \\ z_0 & z_1 & 0 & z_3 & z_4 & z_5 \end{array}\right]$$

* Construct A matrices

- Any o_j is given by the first three entries of the last column of ${}^0T_j = A_1 \dots A_j$ with $o_0 \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = o_1$
- Any z_j is given by

$z_j = {}^0R_j k$

Rotational part of ${}^0T_j$

Thus it is only necessary to compute the matrices ${}^0T_j$ to calculate the Jacobian.

Thus,

$$o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 - c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

$$o_3 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix} = 0 \longrightarrow \text{not zero}$$

and

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \quad z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \quad z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

Returning back to joint velocites and end-effector velocities

$$\underset{\text{end effector velocity}}{\dot{x}} = J(q) \underset{\text{joint velocity}}{\dot{q}}$$

$$\ddot{x} = J(q)\ddot{q} + \left(\frac{d\, J(q)}{dt}\right)\dot{q}$$

Given a vector $\ddot{x}$ of end-effector accelerations, the instantaneous joint acceleration vector $\ddot{q}$ is given as a soln of

$$b = J(q)\ddot{q}$$

where

$$b = \ddot{x} - \left(\frac{d\, J(q)}{dt}\right)\dot{q} \qquad \dot{q} = J(q)^{-1}\dot{x}$$

$$\ddot{q} = J(q)^{-1} b \quad \text{provided that } \det(J(q)) \neq 0$$

STATIC FORCES

$$f = \begin{bmatrix} f_x \\ f_y \\ f_z \\ m_x \\ m_y \\ m_z \end{bmatrix} \qquad \begin{aligned} \underline{f} &= 10i + 0j - 150k \\ \underline{m} &= 0i - 100j + 0k \end{aligned}$$

Transformation of forces between coordinate frames

- Given a force and moment acting at the origin of some coordinate frame attached to a fixed object.
- find the equivalent force and movement acting at, and describe w.r.t some other coordinate frame also attach rigidly to the object.

Use method of virtual work done in the process to solve the problem.

Object is displaced by an imaginary differential amount D from the original position. This is known as a virtual displacement. During virtual displacement a virtual work δW is done.

D is infinitesimally small so that the energy stored in the system does not change. Thus the δW done by a number of forces acting on the object is zero.

Thus $\delta W = F^T D$

$$D = [d_x \;\; d_y \;\; d_z \;\; \delta_x \;\; \delta_y \;\; \delta_z]^T$$

Suppose that this displacement D can be achieved by the application of another force and moment ${}^cF$ acting at some different point on the object, described by a coordinate frame C. Then the same virtual work will result.

$$\delta W = {}^cF^T \, {}^cD$$

$$F^T D = {}^cF^T \, {}^cD$$

define ${}^cD = J \, D$

$$\begin{bmatrix} {}^cd_x \\ {}^cd_y \\ {}^cd_z \\ {}^c\delta_x \\ {}^c\delta_y \\ {}^c\delta_z \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & (p\times n)_x & - & - \\ o_x & - & - & (p\times o)_x & - & - \\ a_x & - & - & (p\times a)_x & - & - \\ 0 & 0 & 0 & n_x & n_y & n_z \\ 0 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

587

(7)

9-12-2002

$$F^T D = {}^cF^T J D$$

$$F^T = {}^cF^T J$$

$$F = J^T\, {}^cF$$

$$\begin{bmatrix} f_x \\ f_y \\ f_z \\ m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} {}^cf_x \\ {}^cf_y \\ {}^cf_z \\ {}^cm_x \\ {}^cm_y \\ {}^cm_z \end{bmatrix}$$

$${}^cf = (J^T)^{-1} F$$

$$= \begin{bmatrix} n_x & n_y & n_z & 0 & 0 & 0 \\ o_x & o_y & o_z & 0 & 0 & 0 \\ a_x & a_y & a_z & 0 & 0 & 0 \\ (p\times n)_x & (p\times n)_y & (p\times n)_z & n_x & n_y & n_z \\ (p\times o)_x & (p\times o)_y & (p\times o)_z & o_x & o_y & o_z \\ (p\times a)_x & (p\times a)_y & (p\times a)_z & a_x & a_y & a_z \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \\ m_x \\ m_y \\ m_z \end{bmatrix}$$

$$\begin{bmatrix} {}^cm_x \\ {}^cm_y \\ {}^cm_z \\ {}^cf_x \\ {}^cf_y \\ {}^cf_z \end{bmatrix} = \begin{bmatrix} & & \\ & J & \\ & & \end{bmatrix} \begin{bmatrix} m_x \\ m_y \\ m_z \\ f_x \\ f_y \\ f_z \end{bmatrix}$$

587 ① 11-12-2002

$$\begin{bmatrix} c_{m_x} \\ c_{m_y} \\ c_{m_z} \\ c_{f_x} \\ c_{f_y} \\ c_{f_z} \end{bmatrix} = J \begin{bmatrix} m_x \\ m_y \\ m_z \\ f_x \\ f_y \\ f_z \end{bmatrix}$$

$c_{m_x} = n((f \times p) + m)$ | $c_{f_x} = n \cdot f$

$c_{m_y} = o((f \times p) + m)$ | $c_{f_y} = o \cdot f$

$c_{m_z} = a((f \times p) + m)$ | $c_{f_z} = a \cdot f$

Example:

Given the coordinate frame

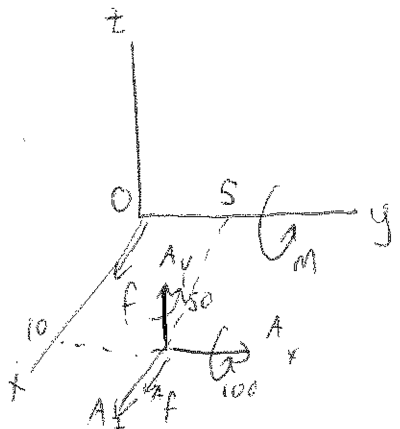
$$A = \begin{array}{c} \begin{array}{cccc} n & o & a & p \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 10 \\ 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

and the following force and moment applied in the base coordinates

$f = 10i + 0j + 0k$

$m = 0i + 100j + 0k$

What is the equivalent force and moment in coordinate frame A?



$^AM_x = n((f \times p) + m)$

$$f \times p = \begin{vmatrix} i & j & k \\ 10 & 0 & 0 \\ 10 & 5 & 0 \end{vmatrix} = 50k$$

$(f \times p) + m = 100j + 50k$

$$^AM_x = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 50 \end{bmatrix} = 100$$

$$^AM_y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 50 \end{bmatrix} = 50$$

$$^AM_z = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \\ 50 \end{bmatrix} = 0$$

$^AM = 100i + 50j + 0k$

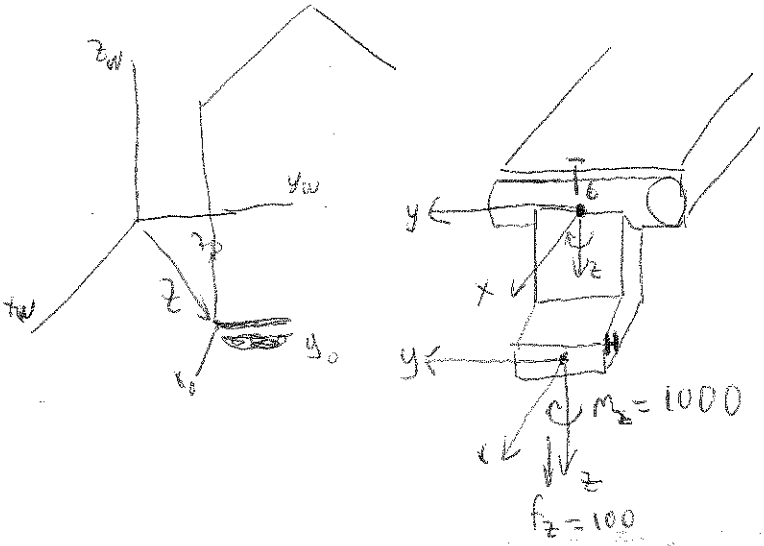
587 | ② | 11-12-2002

$^{A}f_x = n \cdot f = 0$ $\qquad$ $^{A}f = 0i + 0j + 10k$

$^{A}f_y = o \cdot f = 0$

$^{A}f_z = a \cdot f = 10$

<u>Example</u>: A manipulator and end-effector are positioned by $Z\,T_6\,E$ to insert a screw into a hole described by $O.H$ as shown below.



The end-effector is described by

$$E = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The manipulator is to exert a force of

$\underline{f} = 0i + 0j + 100k$

and a moment of

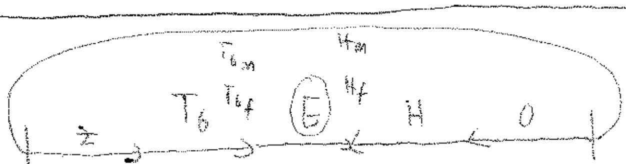
$\underline{m} = 0i + 0j + 1000k$

in hole coordinates, H. What is the equivalent force and moment in T_6?

$Z\,T_6\,E = OH$

$Z\,T_6\,E E^{-1} = OH E^{-1}$

$Z\,T_6 = OH E^{-1}$



$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$p = -2i + 0j - 10k$

$$f \times p = \begin{vmatrix} i & j & k \\ 0 & 0 & 100 \\ -2 & 0 & -10 \end{vmatrix} = -200j$$

$(f \times p) + m = 200j + 1000k$

$$^{T_6}m_x = n \cdot ((f \times p) + m) = [1 \quad 0 \quad 0] \begin{bmatrix} 0 \\ -200 \\ 1000 \end{bmatrix} = 0 \qquad ^{T_6}f_x = n \cdot f = 0$$

$$^{T_6}m_y = o \cdot ((f \times p) + m) = [0 \quad 1 \quad 0] \begin{bmatrix} 0 \\ -200 \\ 1000 \end{bmatrix} = -200 \qquad ^{T_6}f_y = o \cdot f = 0$$

$$^{T_6}m_z = a \cdot ((f \times p) + m) = [0 \quad 0 \quad 1] \begin{bmatrix} 0 \\ -200 \\ 1000 \end{bmatrix} = 1000 \qquad ^{T_6}f_z = a \cdot f = 100$$

$$^{T_6}m = 0i - 200j + 1000k$$

$$^{T_6}f = 0i + 0j + 100k$$

Forces, Moments, and Equivalent Joint Torques

$$\delta W = {}^{T_6}F^T \, {}^{T_6}D = \tau^T Q = [\tau_1 \quad \tau_2 \quad F_3 \quad \tau_4 \quad \tau_5 \quad \tau_6] \begin{bmatrix} d\theta_1 \\ d\theta_2 \\ dd_3 \\ d\theta_4 \\ d\theta_5 \\ d\theta_6 \end{bmatrix}$$

$$\begin{bmatrix} d_x \\ d_y \\ d_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}$$

$$^{T_6}D = J Q$$

$$^{T_6}F^T J Q = \tau^T Q$$

$$^{T_6}F^T J = \tau^T$$

$$\boxed{\tau = J^T \, {}^{T_6}F}$$

← Given

equivalent joint Torques

Example: A stanford manipulator is in a state such that

$$T_6 = \begin{bmatrix} 0 & 1 & 0 & 20 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which corresponds to the following joint coordinates, whose sine and cosine entries are as follows.

Manipulator State

Coordinate	Value	S_i	C_i
θ_1	0	0	1
θ_2	90°	1	0
d_3	$20''$		
θ_4	0	0	1
θ_5	90°	1	0
θ_6	90°	1	0

$$\frac{\partial T_6}{\partial q_i} = \begin{bmatrix} \frac{\partial T_6}{\partial \theta_1} & \frac{\partial T_6}{\partial \theta_2} & \frac{\partial T_6}{\partial d_3} & \frac{\partial T_6}{\partial \theta_4} & \frac{\partial T_6}{\partial \theta_5} & \frac{\partial T_6}{\partial \theta_6} \end{bmatrix}$$

$$\frac{\partial T_6}{\partial \theta_6} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \frac{\partial T_6}{\partial \theta_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ S_6 \\ C_6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial T_6}{\partial \theta_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -S_5 C_6 \\ S_5 S_6 \\ C_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial T_6}{\partial d_3} = \begin{bmatrix} -S_5 C_6 \\ S_5 S_6 \\ C_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial T_6}{\partial \theta_2} = \begin{bmatrix} d_3 (C_4 C_5 C_6 - S_4 S_6) \\ -d_3 (C_4 C_5 C_6 + S_4 C_6) \\ d_3 C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 \\ -S_4 C_5 S_6 + C_4 C_6 \\ S_4 S_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{dT_6}{d\theta_1} = \begin{bmatrix} -d_2[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + s_2d_3(s_4c_5c_6 + c_4s_6) \\ -d_2[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + s_2d_3(-s_4c_5s_6 + c_4c_6) \\ -d_2(c_2c_4s_5 + s_2c_5) + s_2d_3s_4s_5 \\ -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\ s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\ -s_2c_4c_5 + c_2c_5 \end{bmatrix} = \begin{bmatrix} 20 \\ -6 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\frac{\partial T_6}{\partial q_i} = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = J$$

We are to compute joint torques and forces necessary to exert the force and moment

$${}^{T_6}F = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ -200 \\ 1000 \end{bmatrix}$$

587) (i) 18-12-2002

$$\frac{\partial T_6}{\partial q_i} = J = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 \\ -6 & 0 & 1 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{T_6}F = \begin{bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ -200 \\ 1000 \end{bmatrix}$$

$$Z = J^{T}\,{}^{6}f \qquad Z = J^{T}\,{}^{T_6}f$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix} = \begin{bmatrix} & & \\ & J^{T} & \\ & & \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 100 \\ 0 \\ -200 \\ 1000 \end{bmatrix} = \begin{bmatrix} -1000 \\ 2000 \\ 0 \\ -200 \\ 0 \\ 1000 \end{bmatrix}$$

CONTROL OF MANIPULATORS

$$z_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=i}^{n}\sum_{k=1}^{j} tr\left\{\frac{\partial T_j}{\partial q_k} J_j \frac{\partial T_j}{\partial q_i}^{T}\right\}\ddot q_k + \sum_{j=i}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} tr\left\{\frac{\partial T_j}{\partial q_k \partial q_m} J_j \frac{\partial T_j}{\partial q_i}^{T}\right\}\dot q_k \dot q_m$$

$$- \sum_{j=i}^{n} m_j\, g^{T} \frac{\partial T_j}{\partial q_i}\, {}^{j}\bar r_j \qquad \text{for } i = 1, 2, \dots n$$

J_j: Pseudo inertia matrix of link i abt. the i-th coordinate frame

${}^{j}\bar r_j$: position of the center of mass of link i w.r.to the j-th coordinate frame

$$= \sum_{k=1}^{n} D_{ik}\ddot q_k + \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\dot q_k \dot q_m + C_i$$

$$Z(t) = D(q(t))\,\ddot q(t) + h(q(t), \dot q(t)) + C(q(t))$$

$D(q(t))$ $n \times n$ inertial acceleration-related symmetric matrix, with elements:

$$D_{ik} = \sum_{j=\max(i,k)}^{n} tr\left(\frac{\partial T_j}{\partial q_k} J_j \frac{\partial T_j}{\partial q_i}^T\right) \qquad i,k = 1,2,\ldots,n$$

$h(q,\dot q)$: $n \times 1$ coriolis and centrifugal force vector whose elements are

$$h(q,\dot q) = (h_1, h_2, \ldots, h_n)^T$$

$$h_i = \sum_{k=1}^{n}\sum_{m=1}^{n} h_{ikm}\, \dot q_k\, \dot q_m \qquad i=1,2,\ldots,n$$

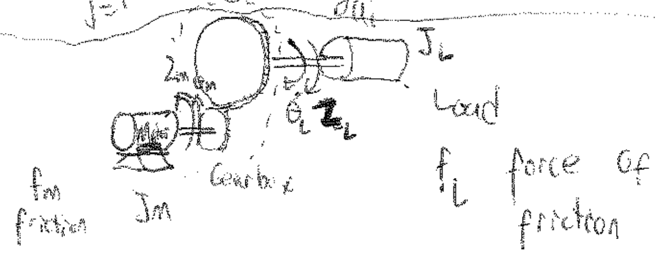
$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} tr\left(\frac{\partial T_j}{\partial q_k \partial q_m} J_j \frac{\partial T_j}{\partial q_i}^T\right)$$

$c(q)$ $n \times 1$ gravity loading force vector

$$C(q) = (c_1, c_2 \ldots c_n)^T$$

$$c_i = \sum_{j=1}^{n} \left(-m_j\, g^T \frac{\partial T_j}{\partial q_i}\, {}^j\bar r_j\right) \qquad i=1,2,\ldots,n$$

Ex:



n = gear ratio

$$\theta_m(s) = \frac{\theta_L(s)}{n}$$

$$Z_L = J_L \ddot\theta_L(t) + f_L \dot\theta_L(t)$$

$$Z_m = J_m \ddot\theta_m(t) + f_m \dot\theta_m(t)$$

$$Z(t) = Z_m(t) + n^2 Z_L$$

$$Z(t) = (J_m + n^2 J_L)\ddot\theta_m(t) + (f_m + n^2 f_L)\dot\theta_m(t)$$

$$= J_{eff}\, \ddot\theta_m(t) + f_{eff}\, \dot\theta_m(t)$$

Effective inertia — effective viscous friction

587

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18-12-2002

$z(t) = K_a\, i_a$ armature current

$V_a(t) = R_a i_a + L_a \frac{di_a}{dt} + e_g$ $\qquad e_g = K_g \dot{\theta}_m(t)$

armature resistance — armature inductance — back emf

$(R_a + L_a s)\; s^2 J_{eff} \theta_m(s) + s f_{eff} \theta_m(s) + K_a K_g s \theta_m(s) = K_a V_a(s)$

$T(s) = s^2 J_{eff} \theta_m(s) + s f_{eff} \theta_m(s)$

$= K_a I_a(s) = K_a \left[\frac{V_a(s) - s K_g \theta_m(s)}{R_a + L_a s} \right]$

$T_a = \frac{L_a}{R_a}$ electrical time constant of the motor armature

$T_m \gg T_a$

T_m = mechanical time constant of the motor

$K \triangleq \frac{K_a}{R_a f_{eff} + K_a K_g}$ motor gain

$\frac{\theta_m(s)}{V_a(s)} = \frac{K_a}{s(s R_a J_{eff} + R_a f_{eff} + K_a K_g)} = \frac{K}{s(T_m s + 1)}$

$T_m \triangleq \frac{R_a J_{eff}}{R_a f_{eff} + K_a K_g}$ motor time constant

$$\frac{\theta_m(s)}{V_a(s)} = \frac{K_a}{s(sR_a J_{eff} + R_a f_{eff} + K_a K_g)} = \frac{K}{s(T_m s + 1)}$$

$$K \triangleq \frac{K_a}{R_a f_{eff} + K_a K_g}$$

$$T_m \triangleq \frac{R_a J_{eff}}{R_a f_{eff} + K_a K_g}$$

$$\frac{\theta_L(s)}{V_a(s)} = \frac{nK}{s(T_m s + 1)}$$

Positional Control for a Single Joint

$$V_a(t) = \frac{K_p\, e(t)}{n} = \frac{K_p(\theta_{L\,desired} - \theta_{L\,actual})}{n} \qquad V_a(s) = \frac{K_p\, E(s)}{n}$$

$$\frac{\theta_L(s)}{E(s)} \triangleq G(s) = \frac{K_a K_p}{s(sR_a J_{eff} + R_a f_{eff} + K_a K_g)}$$

$$\frac{\theta_{L\,act}(s)}{\theta_{L\,des}(s)} \triangleq \frac{G(s)}{1+G(s)} = \frac{K_a K_p / R_a J_{eff}}{s^2 + [(R_a J_{eff} + K_a K_g)/R_a J_{eff}]s + K_a K_g / R_a J_{eff}}$$

In order to increase the system response time and reduce the steady-state error, increase the positional feedback gain and add some damping into the system by adding a derivative of the positional error.

$$V_a(t) = \frac{K_p[\theta_{L\,des} - \theta_{L\,actual}] + K_v[\dot{\theta}_{L\,des} - \dot{\theta}_{L\,act}]}{n} = \frac{K_p\, e(t) + K_v\, \dot{e}(t)}{n}$$

$$\frac{\theta_L(s)}{E(s)} \triangleq G_{PD}(s) = \frac{K_a K_v s + K_a K_p}{s(sR_a J_{eff} + R_a f_{eff} + K_a K_g)}$$

$$\frac{\theta_{L\,act}(s)}{\theta_{L\,des}(s)} \triangleq \frac{G_{PD}(s)}{1+G_{PD}(s)} = \frac{K_a K_v s + K_a K_p}{s^2 R_a J_{eff} + s(R_a f_{eff} + K_a K_g + K_a K_v) + K_a K_p}$$

587 ③ 23-12-2002

In order not to excite the structural oscillation and resonance of the joint, the undamped natural frequency w_n may be set to no more than 0,5 of the structural resonant frequency of the joint, that is

$$w_n \leq 0,5 w_r$$

K_{stiff} : effective stiffness of the joint

$Z_{restoring}$: $K_{stiff}\, \theta_m(t) = Z_{inertia}$ of the motor

$$J_{eff}\, \ddot{\theta}_m(t) + K_{stiff}\, \theta_m(t) = 0$$

$$J_{eff}\, s^2 + K_{stiff} = 0$$

$$w_r = \sqrt{\frac{K_{stiff}}{J_{eff}}}$$ ← is constant

$$\omega_0 = \sqrt{\frac{K_{stiff}}{J_0}}$$

$$\boxed{w_r = w_0 \sqrt{\frac{J_0}{J_{eff}}}}$$

$w_n \leq 0,5 w_r$

$$0 < K_p < \frac{w_0^2 J_0 R_a}{4 K_a}$$

$$K_v \geq \frac{R_a w_0 \sqrt{J_0/J_{eff}} - R_a f_{eff} - K_a K_g}{K_a}$$

Steady-State Errors of the system

$$E(s) = \theta_{L_{des}}(s) - \theta_{L_{act}}(s)$$

i) For a step input of Magnitude A

$$\theta_{L_{des}}(t) = A$$

$$e_{ss}(\text{step}) \triangleq e_{ss_p} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\, \frac{\left[s^2 J_{eff} R_a + s(R_a f_{eff} + K_a K_g)\right]\frac{A}{s} + n R_a D(s)}{s^2 R_a J_{eff} + s(R_a f_{eff} + K_a K_g + K_a K_v) + K_a K_p} \rightarrow \Delta$$

$$= \lim_{s\to 0} s \left[\frac{(n R_a D(s))}{\Delta}\right]$$

$$Z_D(t) = Z_G(t) + Z_C(t) + Z_e$$

$$D(s) = \mathcal{T}_G(s) + \mathcal{T}_C(s) + \frac{\mathcal{T}_e}{s}$$

$$D(q)\ddot{q}(t)+h(q,\dot{q})+C(q_1)=D_a(q_1)\left\{\ddot{q}_1^{d}(t)+K_v\left[\dot{q}^d(t)-\dot{q}(t)\right]+K_p\left[q^d(t)-q(t)\right]\right\}$$
$$+h_a(q,\dot{q})+C(q)$$

if $D_a(q)$, $h_a(q,\dot{q})$ and $C(q)$ are equal to $D(q)$, $h(q,\dot{q})$ and $C(q)$ then the above torque equality reduces to

$$Dq\left(\ddot{e}(t)+K_v\dot{e}(t)+K_p e(t)\right]=0 \quad \text{where } e(t)=q^d(t)-q(t)$$

$D(q)$ is always non-singular. K_p and K_v can be chosen appropriately so the characteristic roots of this equation have negative real parts, then $e(t)$ asymptotically approaches to zero.

Lagrange Euler representation becomes inefficient in the computation of joint torques. Paul suggests that one must not design a closed-loop digital controller using L.E (Lagrange-Euler) representation such as neglecting the velocity related terms of $h_a(q,\dot{q})$ and the off-diagonal elements of the acceleration related matrix $D_a(q)$

Then

$$\tau(t)=\mathrm{diag}\left[D_a(q)\right]\left\{q^d(t)+K_v\left[\dot{q}^d(t)-\dot{q}(t)\right]+K_p\left[q^d(t)-q(t)\right]\right\}+C_a(q)$$

Computer simulations conducted on this control law shows that terms cannot be neglected when the robot arm is moving at high speeds.

587) ① 25-12-2002

Force control, compliness, path trajectory

örnek:
iki robotun
eş zamanlı
birsey'leri tutması

An analogous control law in the joint variable space can be derived from N-E equations of motion to serve a robot arm, The control law is computed recursively using N-E equations. The recursive control law can be obtained by substituting $\ddot{q}_i(t)$ into N-E equations to obtain the necessary joint torque for each actuator.

$$\ddot{q}_i(t) = \ddot{q}_i^d(t) + \sum_{j=1}^{n} K_v^{ij}\left[\dot{q}_j^d(t) - \dot{q}_j(t)\right] + \sum_{j=1}^{n} K_p^{ij}\left[q_j^d(t) - q_j(t)\right]$$

1. The first term generates desired torque for each joint if there is no modelling error and the physical system parameters are known, However there are errors due to backlash gear friction uncertainity about the inertia parameters and time delay into the servo loop so that deviation from the desired joint trajectory will be inevitable.

2. The remaining terms in N.E eqns of motion will generate the correction torques, to compansate for small deviations from the desired ~~trajectory~~ joint trajectory.

Motion Controls

1. Joint Motion Control
   - 1.1 Joint servo mechanism
   - 1.2 Computed torque technique
   - 1.3 Minimum time control
   - 1.4 Variable structure control
   - 1.5 Nonlinear decoupled control
2. Resolved Motion Controls
   - 2.1 Resolved motion rate Control
   - 2.2 Resolved " acceleration control

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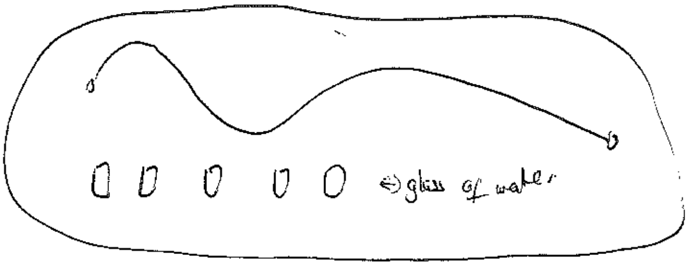
Trajectory planning

(1) GROSS MOTION
(2) FINE MOTION



Trans Rot(z,ϕ) Rot(y,θ)

(Need two rotation vectors to carry the glass together with the robot arm)

$$ {}^{0}T_{6}\ {}^{6}T_{Tool} = {}^{0}C_{working}(t)\ {}^{working}P_{obj} $$

4x4 4x4 4x4 4x4

Describing

the manipulator hand position + orientation w.r.t the base coordinate frame

the tool position and orientation w.r.t hand coordinate frame

It describes the tool end-point whose motion to be controlled.

working coordinate frame of the object w.r.t the base coordinate frame as a function of time

The desired gripping position (or point of manipulation) of the object for the end-effector wrt the working coordinate frame

$$ \boxed{{}^{0}T_{6} = C_{working}(t)\ {}^{w}P_{obj} \left({}^{6}T_{Tool}\right)^{-1}} $$

$$ {}^{0}T_{6}\left({}^{6}T_{Tool}\right)_{1} = \left[C_{w}(t)\right]_{1}\left({}^{w}P_{obj}\right)_{1} $$

$$ {}^{0}T_{6}\left({}^{6}T_{Tool}\right)_{2} = \left[C_{w}(t)\right]_{2}\left({}^{w}P_{obj}\right)_{2} $$

$$ \vdots $$

$$ {}^{0}T_{6}\left({}^{6}T_{Tool}\right)_{N} = \left[C_{w}(t)\right]_{N}\left({}^{w}P_{obj}\right)_{N} $$

$$ \boxed{\begin{aligned} T_{6}\ {}^{Tool}T_{1} &= C_{1}(t)\ P_{1} \\ T_{6}\ {}^{Tool}T_{2} &= C_{2}(t)\ P_{2} \\ &\vdots \\ T_{6}\ {}^{Tool}T_{N} &= C_{N}(t)\ P_{N} \end{aligned}} $$

587

II

30-12-2002

From the position defined by $C_i(t)\ P_i$ we can obtain the distance between consecutive points. If we are given linear v and angular ω velocities we can obtain the time ~~required~~ requested t_i to move from position i to $i+1$

$$T_6\ {}^{Tool}T_1 = C_1(t)\ P_{i1}$$

$$T_6\ {}^{Tool}T_2 = C_2(t)\ P_{i2}$$

$$\boxed{P_{i2} = C_2^{-1}(t)\ C_1(t)\ P_{i1}\left({}^{Tool}T_2\right)^{-1}}$$

$D(\lambda) \qquad 0 \leq t = \frac{t}{T} \leq 1$

Driving function

At position i $\quad t=0 \quad \lambda=0 \quad D(0) \quad 4\times 4$

$$P_{i,i+1} = P_i\ D(1) \quad * \qquad D(1) = \left(P_i^{-1}\ P_{i,i+1}\right)$$

$$P_{i,i} = P_i\ D(0)$$

$$P_{i,1} \triangleq A = \begin{bmatrix} n_A & o_A & a_A & p_A \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad P_{i,i+1} \triangleq B = \begin{bmatrix} n_B & o_B & a_B & p_B \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x^A & o_x^A & a_x^A & p_x^A \\ n_y^A & o_y^A & a_y^A & p_y^A \\ n_z^A & o_z^A & a_z^A & p_z^A \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} n_x^B & o_x^B & a_x^B & p_x^B \\ n_y^B & o_y^B & a_y^B & p_y^B \\ n_z^B & o_z^B & a_z^B & p_z^B \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D(1) = \begin{bmatrix} n_A \cdot n_B & n_A \cdot o_B & n_A \cdot a_B & n_A\,(p_B - p_A) \\ o_A \cdot n_B & o_A \cdot o_B & o_A \cdot a_B & o_A\,(p_B - p_A) \\ a_A \cdot n_B & a_A \cdot o_B & a_A \cdot a_B & a_A\,(p_B - p_A) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If λ varies linearly with t then the resultant motion represented by $D(\lambda)$ will correspond to a constant linear velocity and angular velocities. The translational motion can be represented by

$$L(\lambda) = \begin{bmatrix} 1 & 0 & 0 & \lambda_x \\ 0 & 1 & 0 & \lambda_y \\ 0 & 0 & 1 & \lambda_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the motion will be along the straight line joining $P_{i,i}$ to $P_{i,i+1}$

The first rotational motion can be represented by

$$R_A(\lambda) = \begin{bmatrix} s^2\psi V(\lambda,\theta) + C(\lambda\theta) & -s\psi C\psi V(\lambda\theta) & C\psi S(\lambda\theta) & 0 \\ -s\psi C\psi C(\lambda\theta) & [C^2\psi V(\lambda\theta) + C(\lambda\theta)] & s\psi S(\lambda\theta) & 0 \\ -C\psi S(\lambda\theta & -S\psi S(\lambda\theta) & C(\lambda\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and it rotates the approach vector at $P_{i,i}$ to the approach vector at P_i

The second rotational motion represented by $R_B(\lambda)$ rotates the orientation vector at $P_{i,i}$ into the orientation vector at $P_{i,i+1}$ about the tool axis.

$V(\lambda\theta) = \text{Versine}(\lambda\theta)$
$= 1 - \cos(\lambda\theta)$

$$R_B(\lambda) = \begin{bmatrix} C(\lambda\phi) & -S(\lambda\phi) & 0 & 0 \\ S(\lambda\phi) & C(\lambda\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_A(\lambda)$ is a rotation of an angle θ about the orientation vector of $P_{i,i}$ which is rotated an angle of ψ about the approach vector.

$R_B(\partial)$ is a rotation of an angle ϕ about the approach vector of the tool at $P_{i,i+1}$

$$D(\partial) = \begin{bmatrix} d_n & d_o & d_a & d_p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$d_o = \begin{bmatrix} -s(\partial\phi)\,[s^2\psi V(\partial\theta) + c(\partial\theta)] + c(\partial\theta)\,[-s\psi c\psi\, V(\partial\theta)] \\ -s(\partial\phi)\,[-s\psi\, c\psi\, V(\partial\theta)] + c(\partial\theta)\,[c^2\psi V(\partial\theta) + c(\partial\theta)] \\ -s(\partial\phi)\,[-c\psi\, s(\partial\theta)] + c(\partial\theta)\,[-s\psi\, s(\partial\theta)] \end{bmatrix}$$

$$d_a = \begin{bmatrix} c\psi\, s(\partial\theta) \\ s\psi\, s(\partial\theta) \\ c(\partial\theta) \end{bmatrix} \qquad d_p = \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \qquad d_n = d_o \times d_a$$

$$L(\partial) = D(\partial)\, R_B^{-1}(\partial)\, R_A^{-1}(\partial)$$

$$x = n_A\,(P_B - P_A)$$
$$y = o_A\,(P_B - P_A)$$
$$z = a_A\,(P_B - P_A)$$

$$R_a(\partial) = L^{-1}(\partial)\, D(\partial)\, R_B^{-1}(\partial)$$

$$\psi = \tan^{-1}\left(\frac{o_A \cdot d_B}{n_A \cdot d_B}\right) \qquad -\pi \leq \psi \leq \pi$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{(n_A a_B)^2 + (o_A \cdot a_B)^2}}{a_A\, a_B}\right] \qquad 0 \leq \theta \leq \pi$$

(IV)

30-12-2002

$R_B(\partial) = P_A^{-1}(\partial)\ C^{-1}(\partial)\ D(\partial)$

$s\phi = -s\psi\, c\psi\, v(\partial\theta)\ (n_A \cdot n_B) + [c^2\psi\, v(\partial\theta) + c(\partial\theta)]\ (o_A \cdot o_B) - s\psi\, s(\partial\theta)(a_A \cdot n_B)$

$c\phi = -s\psi\, c\psi\, v(\partial\theta)\ (n_A \cdot o_B) + [c^2\psi\, v(\partial\theta) + c(\partial\theta)]\ (o_A \cdot o_B) - s\psi\, s(\partial\theta)(a_A \cdot o_B)$

$$\phi = \tan^{-1}\left[\frac{s\phi}{c\phi}\right] \qquad -\pi \leq \phi \leq \pi$$