DYNAMICS

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ÇANKAYA ÜNİVERSİTESİ
MEKATRONİK MÜHENDİSLİĞİ BÖLÜMÜ

Introduction

- Kinematic equations describe the motion of the robot without consideration of the forces and torques producing the motion.
- The dynamic equations explicitly describe the relationship between **force** and **motion**.

The equations of motion are important to consider in
- design of robots,
- simulation and animation of robot motion, and
- design of control algorithms.

- provides the relation between

generalized forces $u(t)$ acting on the robot

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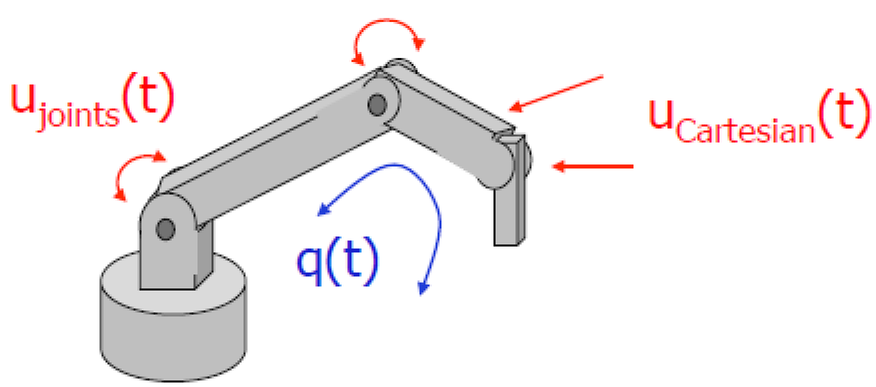
robot motion, i.e.,
assumed configurations $q(t)$ over time



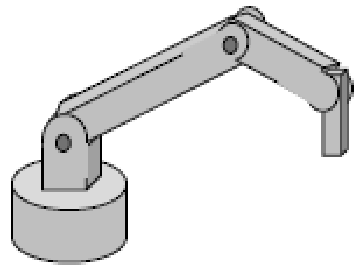
a system of 2nd order differential equations

$$\Phi(q,\dot{q},\ddot{q}) = u$$

Direct Dynamics

- direct relation

$$u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \rightarrow$$



$$\rightarrow \quad q(t) = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

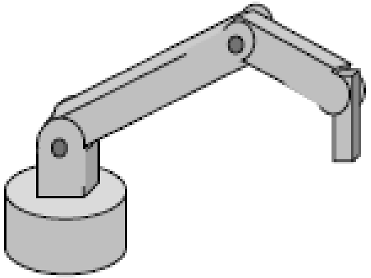
input for $t \in [0,T]$ + $q(0), \dot{q}(0)$

initial state at $t = 0$

- experimental solution
  - apply torques/forces with motors and measure joint variables with encoders (with sampling time T_c)
- solution by simulation ⟷ $\Phi(q,\dot{q},\ddot{q}) = u$
  - use dynamic model and integrate numerically the differential equations (with simulation step $T_s \leq T_c$)

Inverse Dynamics

- inverse relation

$q_d(t), \dot{q}_d(t), \ddot{q}_d(t)$ →  → $u_d(t)$

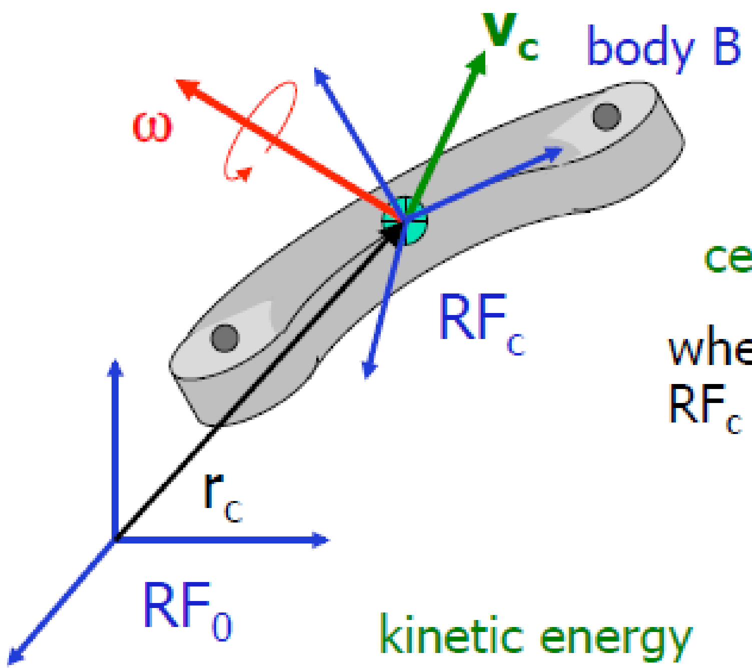
desired motion for $t \in [0,T]$ — required input for $t \in [0,T]$

- experimental solution
  - repeated motion trials of direct dynamics using $u_k(t)$, with iterative learning of nominal torques updated on trial k+1 based on the error in [0,T] measured in trial k: $u_k(t) \Rightarrow u_d(t)$
- analytic solution ⟷ $\Phi(q,\dot{q},\ddot{q}) = u$
  - use dynamic model and compute algebraically the values $u_d(t)$ at every time instant t

Introduction

- Kinematic equations describe the motion of the robot without consideration of the forces and torques producing the motion.
- The dynamic equations explicitly describe the relationship between **force** and **motion**.
- We introduce the so-called **Euler-Lagrange equations**
- In order to determine the **Euler-Lagrange equations** in a specific situation, one has to form the **Lagrangian** of the system, which is the difference between the **kinetic energy** and the **potential energy**;

Kinetic energy of a rigid body



mass density

mass $\quad m = \int_B \rho(x,y,z)\,dx\,dy\,dz = \int_B dm$

position of center of mass (CoM) $\quad r_c = \frac{1}{m}\int_B r\,dm$

when all vectors are referred to a body frame RF_c attached to the CoM, then

$$r_c = 0 \quad \Rightarrow \quad \int_B r\,dm = 0$$

kinetic energy $\quad T = \frac{1}{2}\int_B v^T(x,y,z)\,v(x,y,z)\,dm$

(fundamental) kinematic relation for a rigid body $\quad v = v_c + \omega \times r = v_c + S(\omega)r$

skew-symmetric matrix

$$T = \frac{1}{2}\int_B [v_c + S(\omega) r]^T [v_c + S(\omega) r] dm$$

$$= \frac{1}{2}\int_B v_c^T v_c \, dm + \int_B v_c^T S(\omega) r \, dm + \frac{1}{2}\int_B r^T S^T(\omega) S(\omega) r \, dm$$

sum of elements on the diagonal of a matrix

$a^T b = \text{trace}\{a b^T\}$

$$= \frac{1}{2} m \, v_c^T v_c$$

$$= v_c^T S(\omega) \int_B r \, dm = 0$$

$$= \frac{1}{2}\int_B \text{trace}\{S(\omega) r \cdot r^T S^T(\omega)\} dm$$

$$= \frac{1}{2} \text{trace}\left\{S(\omega)\left(\int_B r \cdot r^T dm\right) S^T(\omega)\right\}$$

$$= \frac{1}{2} \text{trace}\{S(\omega) \, J_c \, S^T(\omega)\}$$

Euler matrix

$$= \frac{1}{2} \omega^T I_c \, \omega$$

translational kinetic energy (point mass in CoM)

+

rotational kinetic energy (of the whole body)

body inertia matrix (around the CoM)

König theorem

Ex #1: provide the expressions of the elements of Euler matrix J_c

Ex #2: prove last equality and provide the expressions of the elements of inertia matrix I_c

The Inertia Tensor

- The inertia matrix expressed in the body attached frame is a constant matrix independent of the motion of the object and easily computed.

$$\mathcal{I} = R I R^T$$

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\begin{aligned}
I_{xx} &= \int\int\int (y^2 + z^2)\rho(x,y,z)dx\ dy\ dz \\
I_{yy} &= \int\int\int (x^2 + z^2)\rho(x,y,z)dx\ dy\ dz \\
I_{zz} &= \int\int\int (x^2 + y^2)\rho(x,y,z)dx\ dy\ dz \\
I_{xy} = I_{yx} &= -\int\int\int xy\rho(x,y,z)dx\ dy\ dz \\
I_{xz} = I_{zx} &= -\int\int\int xz\rho(x,y,z)dx\ dy\ dz \\
I_{yz} = I_{zy} &= -\int\int\int yz\rho(x,y,z)dx\ dy\ dz
\end{aligned}$$

Kinetic Energy for an n-Link Robot

$$\sum_j d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k$$

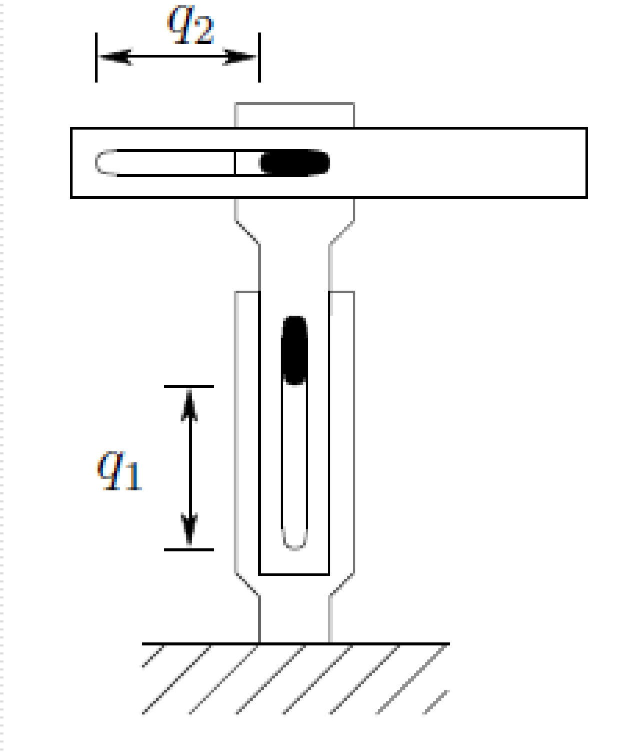
$$c_{ijk} \quad := \quad \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$$

$$\phi_k \quad = \quad \frac{\partial P}{\partial q_k}$$

Matrix Form of Euler-Lagrange Equations

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) \quad = \quad \tau$$

Two-link cartesian robot

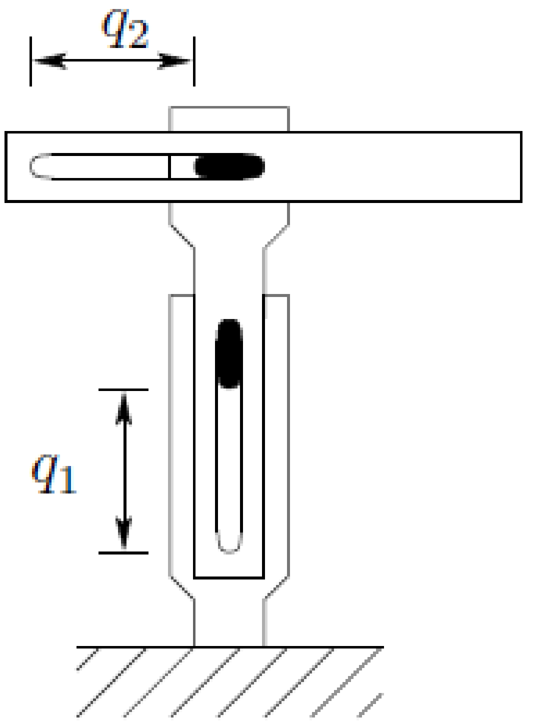


$k = 1, \ldots, n$

Euler-Lagrange equations

$$\sum_j d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k$$

Two-link cartesian robot

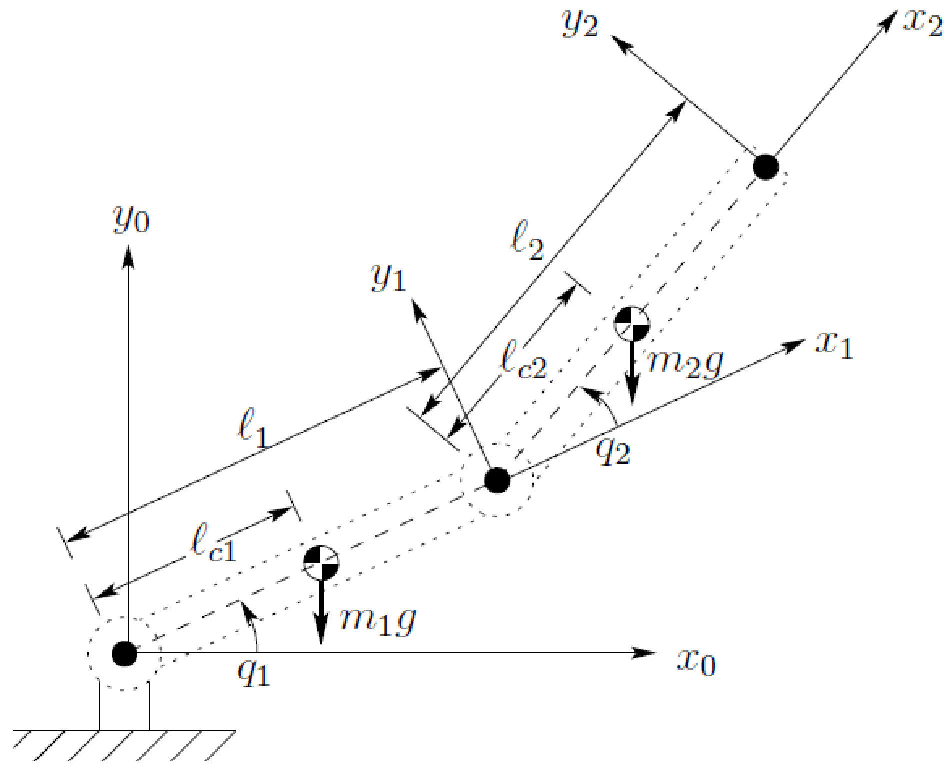


$$
\begin{aligned}
(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) &= f_1 \\
m_2\ddot{q}_2 &= f_2
\end{aligned}
$$

Planar Elbow Manipulator

$$v_{c1} = J_{v_{c1}} \dot{q}$$

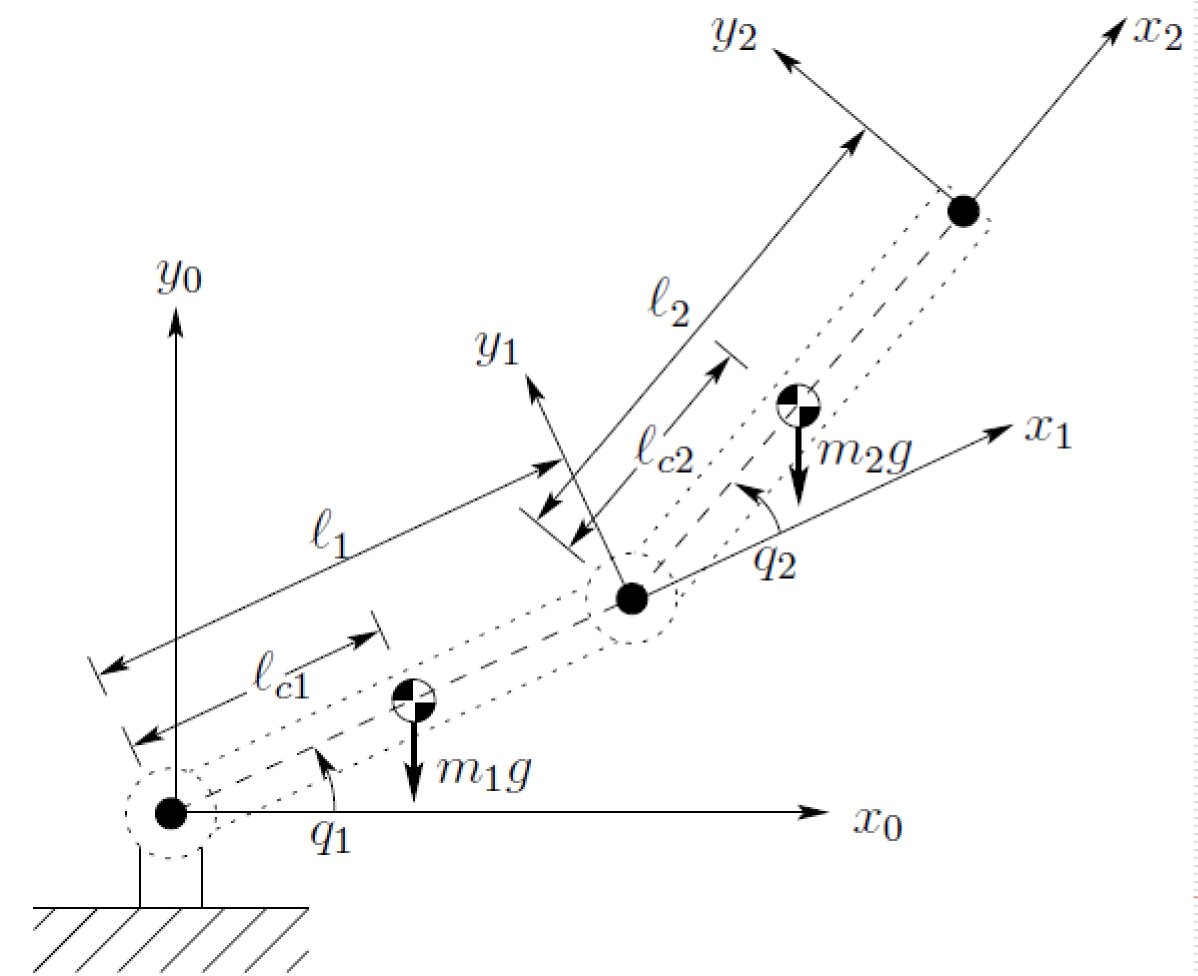
$$J_{v_{c1}} = \begin{bmatrix} -\ell_c \sin q_1 & 0 \\ \ell_{c1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$



$$v_{c2} = J_{v_{c2}} \dot{q}$$

$$J_{v_{c2}} = \begin{bmatrix} -\ell_1 \sin q_1 - \ell_{c2} \sin(q_1 + q_2) & -\ell_{c2} \sin(q_1 + q_2) \\ \ell_1 \cos q_1 + \ell_{c2} \cos(q_1 + q_2) & \ell_{c2} \cos(q_1 + q_2) \\ 0 & 0 \end{bmatrix}$$

Planar Elbow Manipulator



$$D(q) = m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} + \begin{bmatrix} I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix}$$

$$\begin{aligned} d_{11} &= m_1 \ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1 \ell_{c2}^2 + 2\ell_1 \ell_{c2} \cos q_2) + I_1 + I_2 \\ d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1 \ell_{c2} \cos q_2) + I_2 \\ d_{22} &= m_2 \ell_{c2}^2 + I_2 \end{aligned}$$

$$c_{ijk} := \frac{1}{2}\left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\}$$

$$\begin{aligned} c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\ c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2 \ell_1 \ell_{c2} \sin q_2 =: h \\ c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\ c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\ c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\ c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0 \end{aligned}$$

$$\phi_k = \frac{\partial P}{\partial q_k}$$

$$P_1 = m_1 g \ell_{c1} \sin q_1$$

$$P_2 = m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2))$$

$$P = P_1 + P_2 = (m_1 \ell_{c1} + m_2 \ell_1) g \sin q_1 + m_2 \ell_{c2} g \sin(q_1 + q_2)$$

$$\phi_1 = \frac{\partial P}{\partial q_1} = (m_1 \ell_{c1} + m_2 \ell_1) g \cos q_1 + m_2 \ell_{c2} g \cos(q_1 + q_2)$$

$$\phi_2 = \frac{\partial P}{\partial q_2} = m_2 \ell_{c2} \cos(q_1 + q_2)$$

Euler-Lagrange equations

$$\sum_j d_{kj}(q)\ddot{q}_j + \sum_{i,j} c_{ijk}(q)\dot{q}_i\dot{q}_j + \phi_k(q) = \tau_k \quad k = 1, \ldots, n$$

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + \phi_1 = \tau_1$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + \phi_2 = \tau_2$$