Forward And Inverse Kinematics

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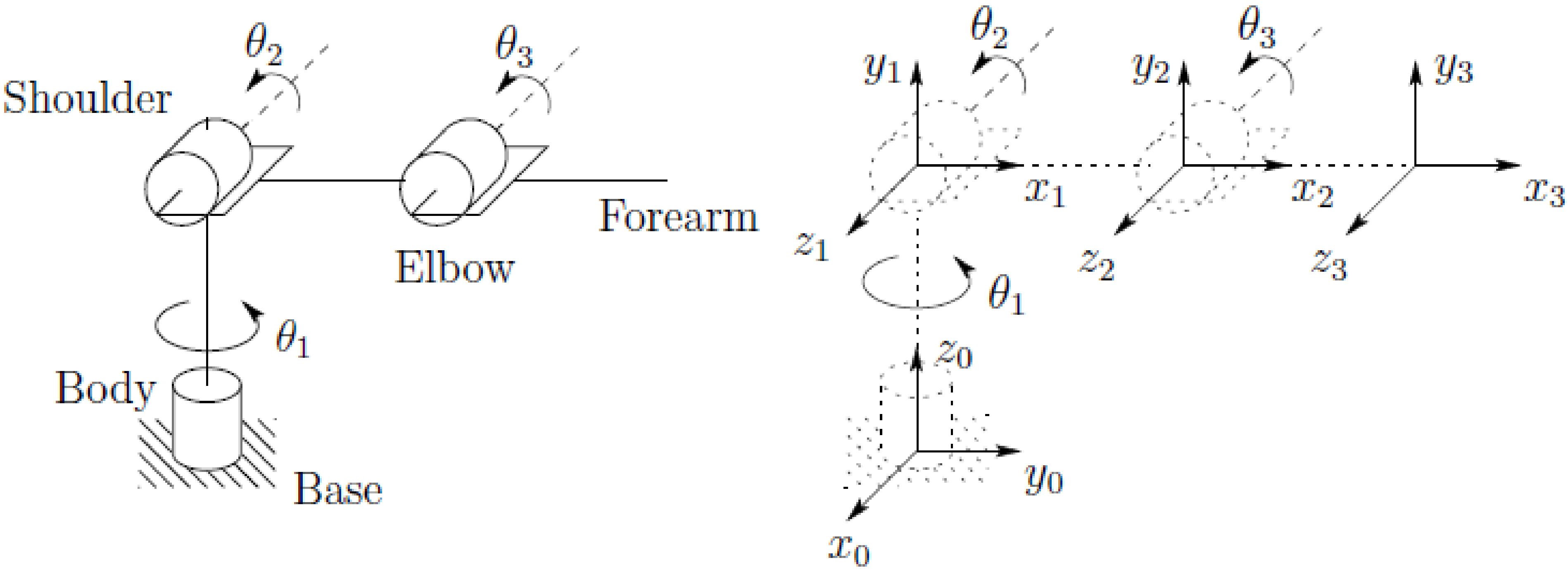
ÇANKAYA ÜNİVERSİTESİ
MEKATRONİK MÜHENDİSLİĞİ BÖLÜMÜ

Introduction

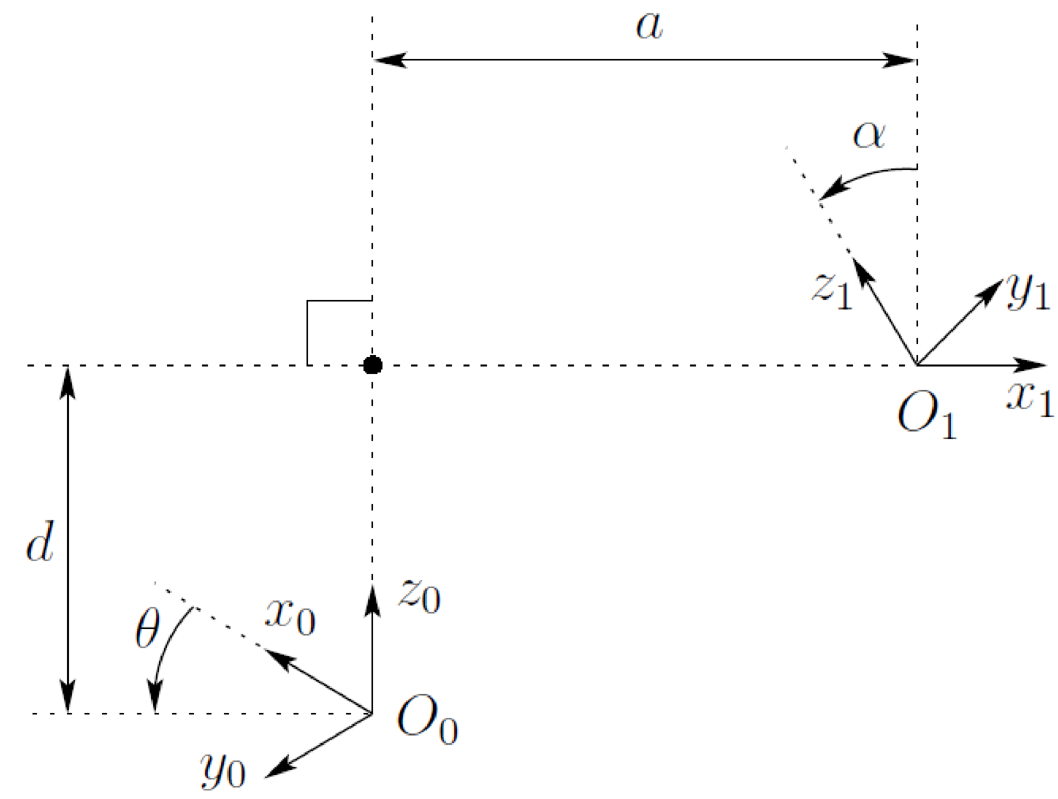
- In this chapter we consider the **forward** and **inverse** kinematics for serial link manipulators.
- Kinematics describes the motion of the manipulator without consideration of the forces and torques causing the motion.
- Forward kinematics:
  - joint variables -> position and orientation of the end-effector
- Inverse kinematics:
  - position and orientation of the end-effector -> joint variables

Figure 3.1: A Coordinate Frame is attached rigidly to each **link**



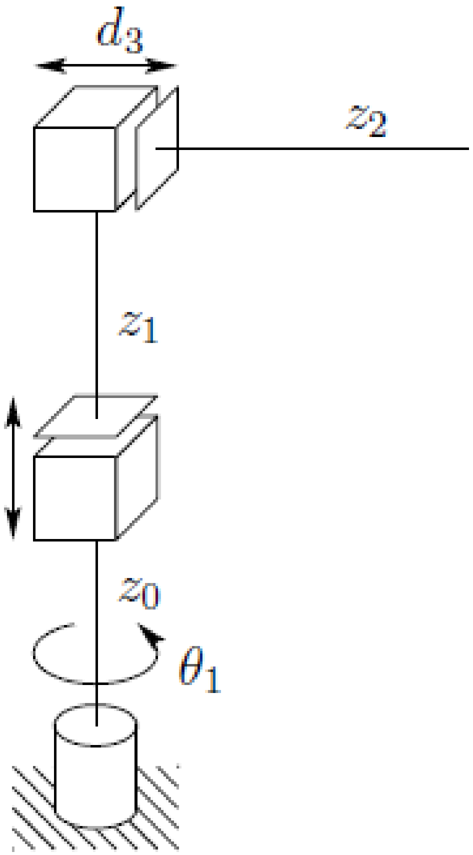
$$
\begin{aligned}
A_i &= Rot_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}Rot_{x,\alpha_i} \qquad (3.10)\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_ic_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_is_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Denavit Hartenberg Representation



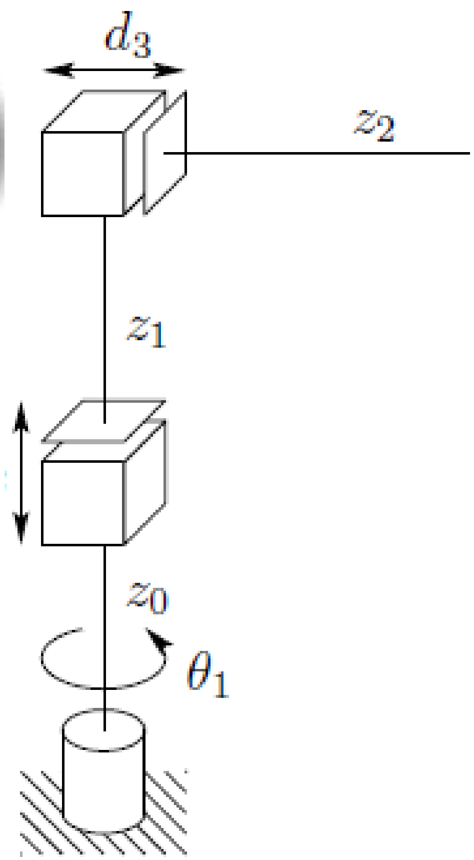
Three-link cylindrical manipulator

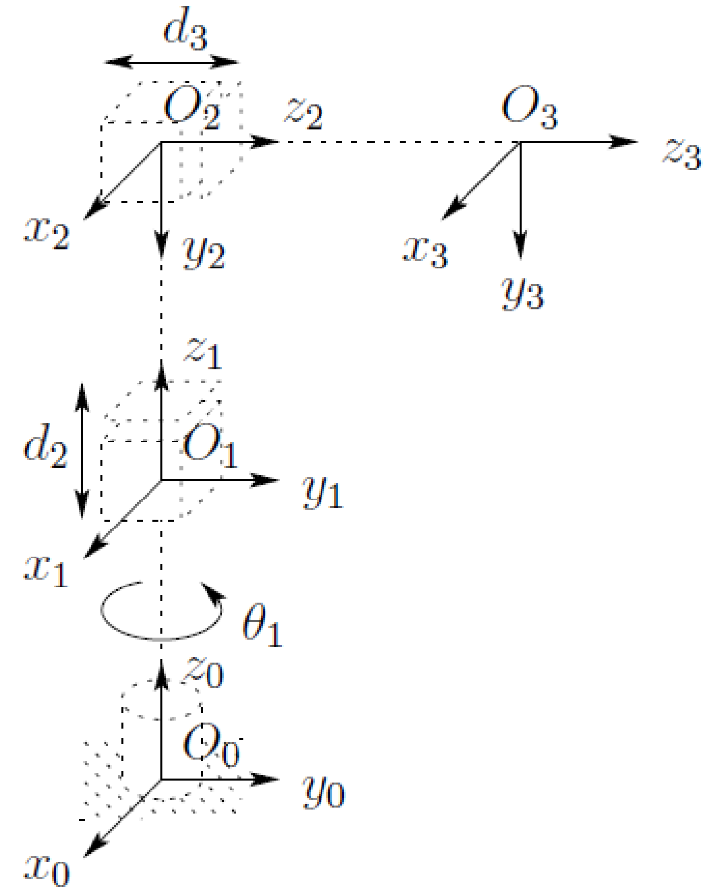
The Seiko RT3300 Robot



Three-link cylindrical manipulator

The Seiko RT3300 Robot



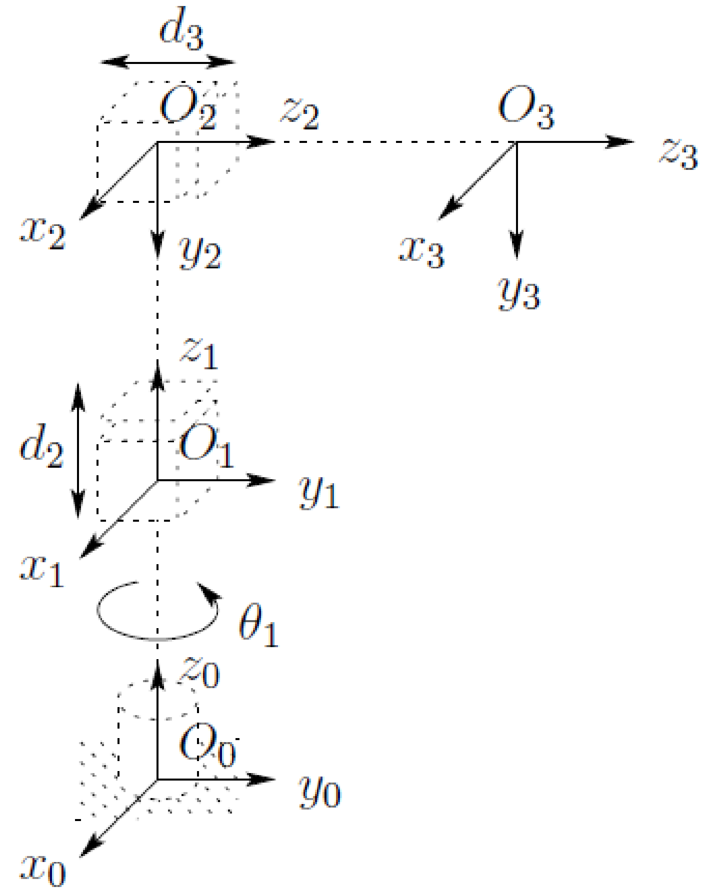


Three-link cylindrical manipulator

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Procedure Based On The D-H Convention

Step 1: Locate and label the joint axes $z_0, \ldots, z_{n-1}$.

Step 2: Establish the base frame. Set the origin anywhere on the z_0-axis. The x_0 and y_0 axes are chosen conveniently to form a right-hand frame.

For $i = 1, \ldots, n-1$, perform Steps 3 to 5.

Step 3: Locate the origin O_i where the common normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1} locate O_i at this intersection. If z_i and z_{i-1} are parallel, locate O_i in any convenient position along z_i.

Step 4: Establish x_i along the common normal between z_{i-1} and z_i through O_i, or in the direction normal to the $z_{i-1} - z_i$ plane if z_{i-1} and z_i intersect.

Step 5: Establish y_i to complete a right-hand frame.

Step 6: Establish the end-effector frame $o_n x_n y_n z_n$. Assuming the n-th joint is revolute, set $z_n = \boldsymbol{a}$ along the direction z_{n-1}. Establish the origin O_n conveniently along z_n, preferably at the center of the gripper or at the tip of any tool that the manipulator may be carrying. Set $y_n = \boldsymbol{s}$ in the direction of the gripper closure and set $x_n = \boldsymbol{n}$ as $\boldsymbol{s} \times \boldsymbol{a}$. If the tool is not a simple gripper set x_n and y_n conveniently to form a right-hand frame.

Step 7: Create a table of link parameters a_i, d_i, α_i, θ_i.

a_i = distance along x_i from O_i to the intersection of the x_i and z_{i-1} axes.

d_i = distance along z_{i-1} from O_{i-1} to the intersection of the x_i and z_{i-1} axes. d_i is variable if joint i is prismatic.

α_i = the angle between z_{i-1} and z_i measured about x_i (see Figure 3.3).

θ_i = the angle between x_{i-1} and x_i measured about z_{i-1} (see Figure 3.3). θ_i is variable if joint i is revolute.

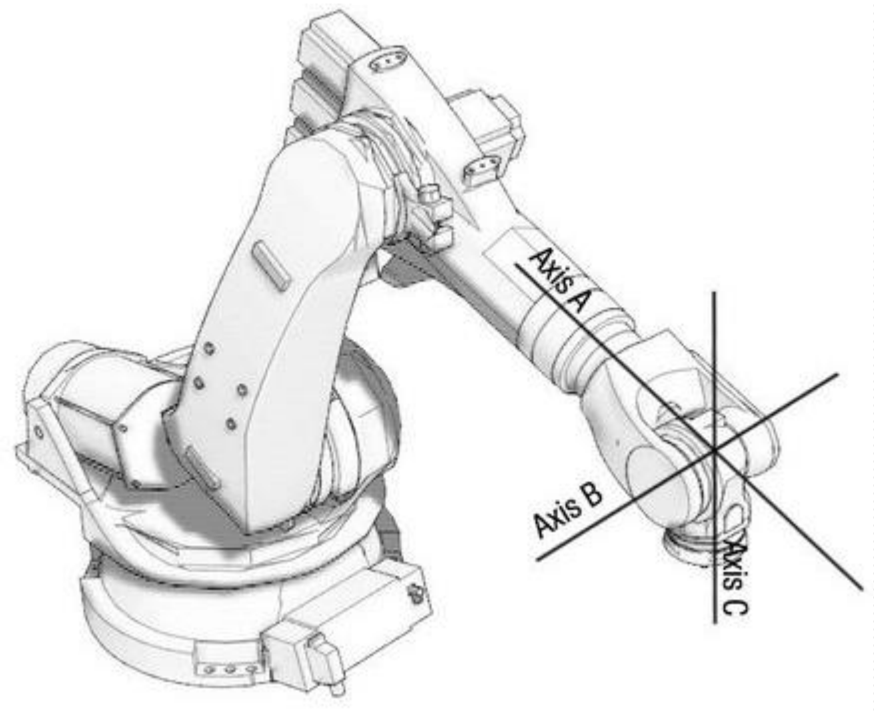
Step 8: Form the homogeneous transformation matrices A_i by substituting the above parameters into (3.10).

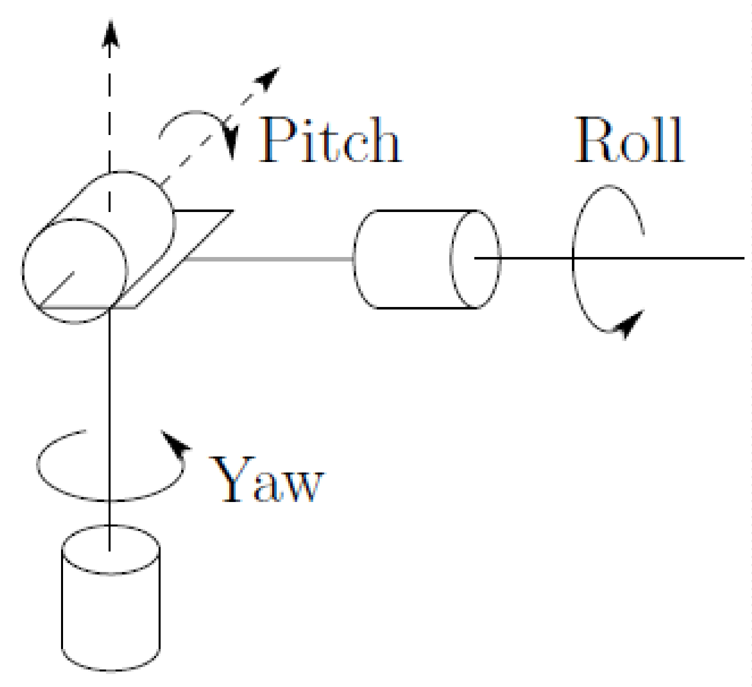
Step 9: Form $T_n^0 = A_1 \cdots A_n$. This then gives the position and orientation of the tool frame expressed in base coordinates.

Ambiguities in defining DH frames

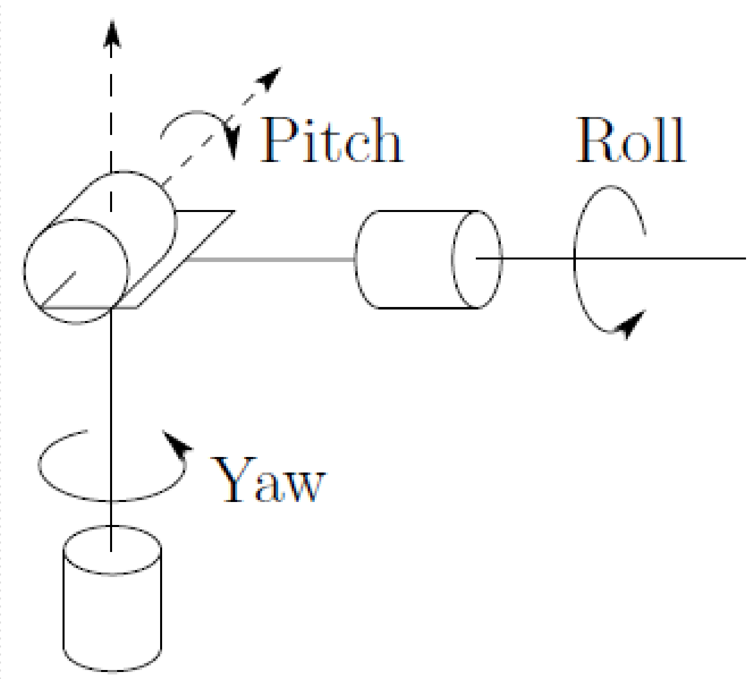
- *frame$_0$:* origin and x_0 axis are arbitrary
- *frame$_n$:* z_n axis is not specified (but x_n must be orthogonal to and intersect z_{n-1})
- *positive* direction of z_{i-1} (up/down on joint i) is arbitrary
  - choose one, and try to avoid "flipping over" to the next one
- *positive* direction of x_i (on axis of link i) is arbitrary
  - we often take $x_i = z_{i-1} \times z_i$ when successive joint axes are incident
  - when natural, we follow the direction "from base to tip"
- if z_{i-1} and z_i are *parallel:* common normal not uniquely defined
  - O_i is chosen arbitrarily along z_i, but try to "zero out" parameters
- if z_{i-1} and z_i are *coincident:* normal x_i axis may be chosen at will
  - again, we try to use "simple" constant angles $(0, \pi/2)$
  - this case may occur only if the two joints are of different kind (P & R)

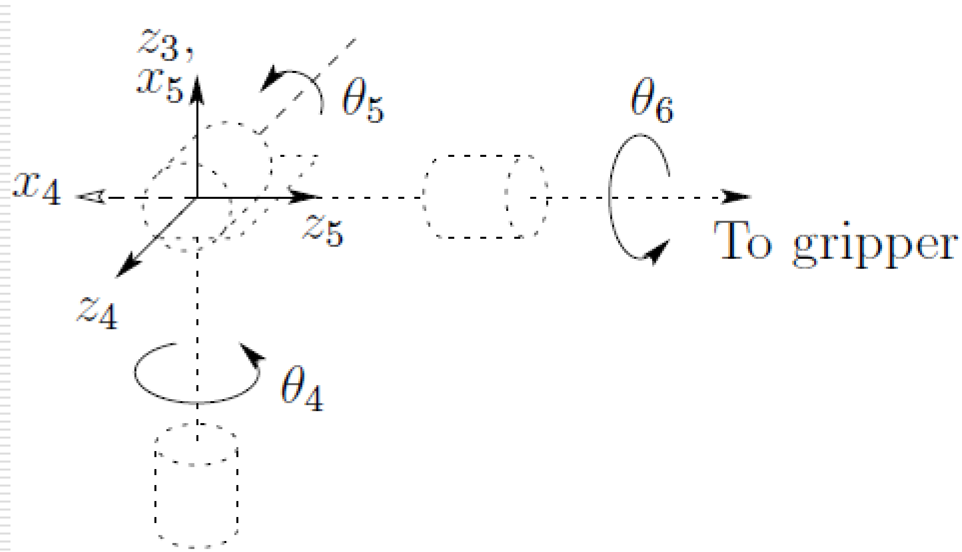
Spherical Wrist



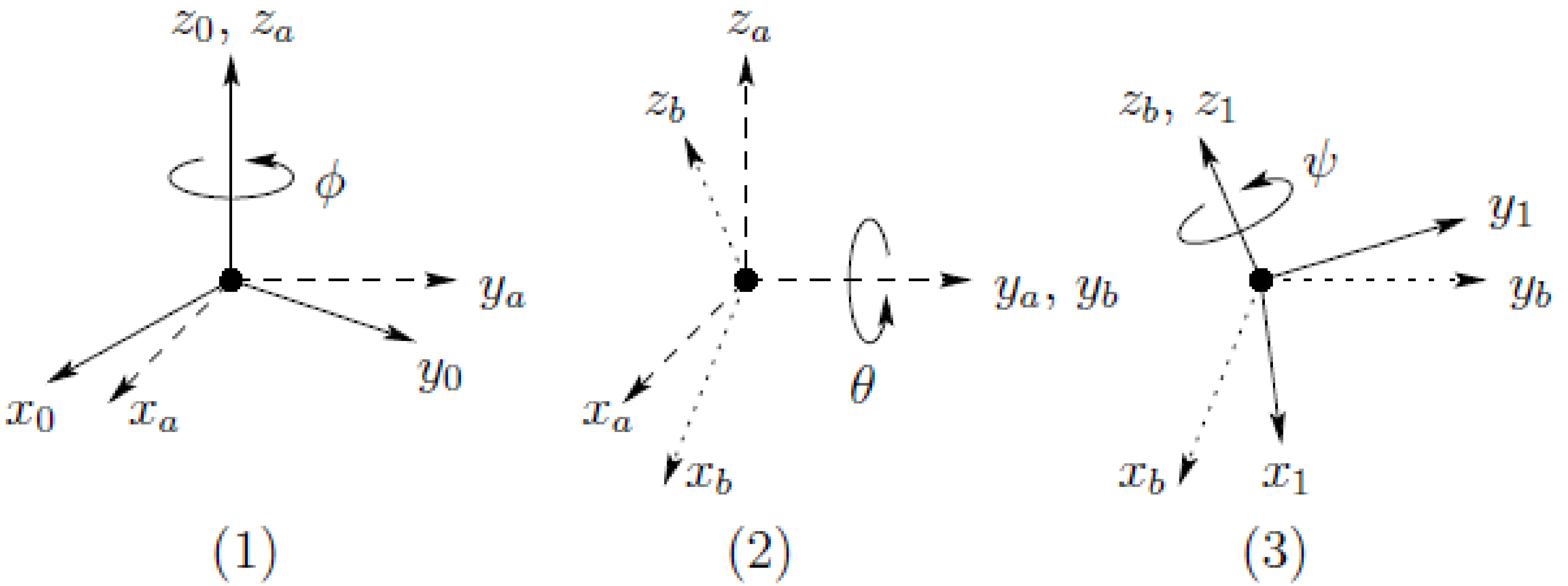


Spherical Wrist





Euler Angle Representation



$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

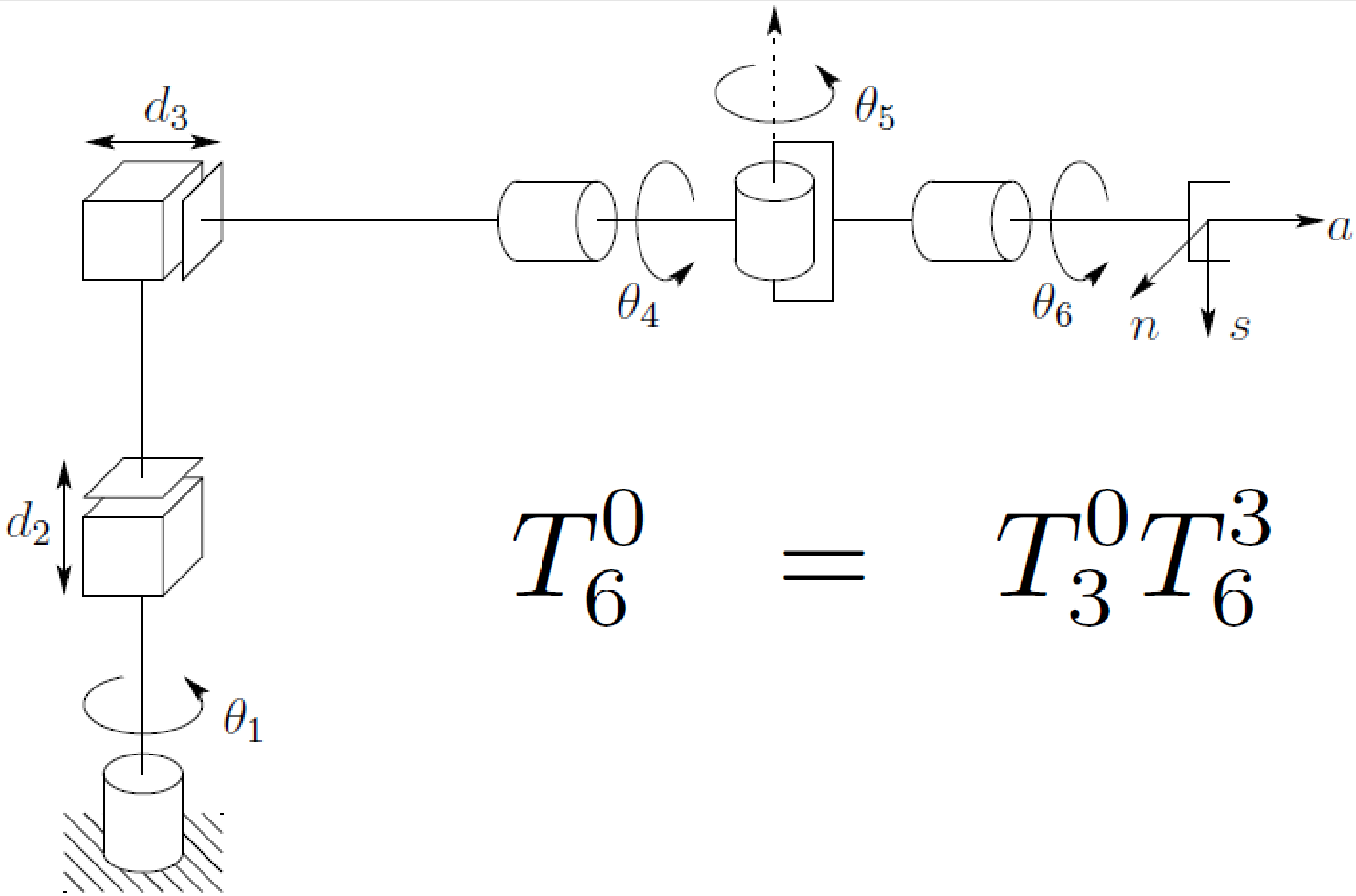
Euler Angle Representation

$$
\begin{aligned}
R_{ZYZ} &= R_{z,\phi} R_{y,\theta} R_{z,\psi} \\
&= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}
\end{aligned}
$$

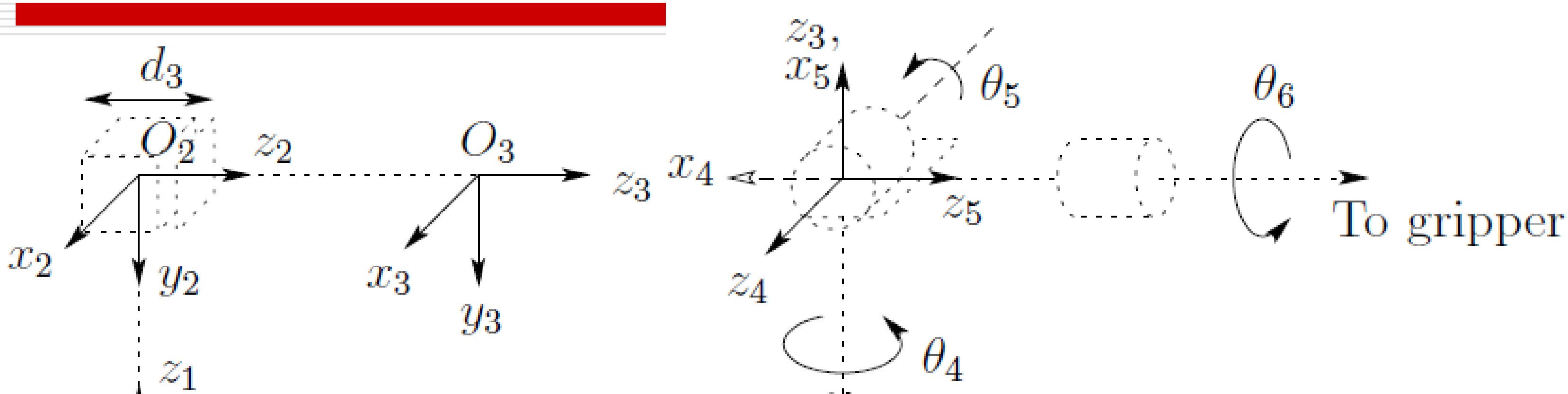
Spherical Wrist

$$
\begin{aligned}
T_6^3 &= A_4 A_5 A_6 \\
&= \begin{bmatrix} R_6^3 & o_6^3 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

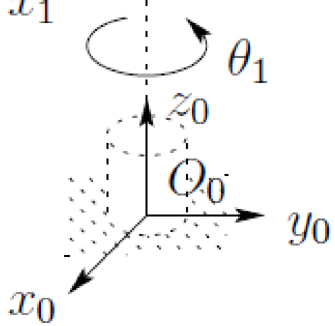
Cylindrical Manipulator with Spherical Wrist



$$T_6^0 = T_3^0 T_6^3$$



$$T_6^0 = T_3^0 T_6^3$$



$$T_3^0 = A_1 A_2 A_3 = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

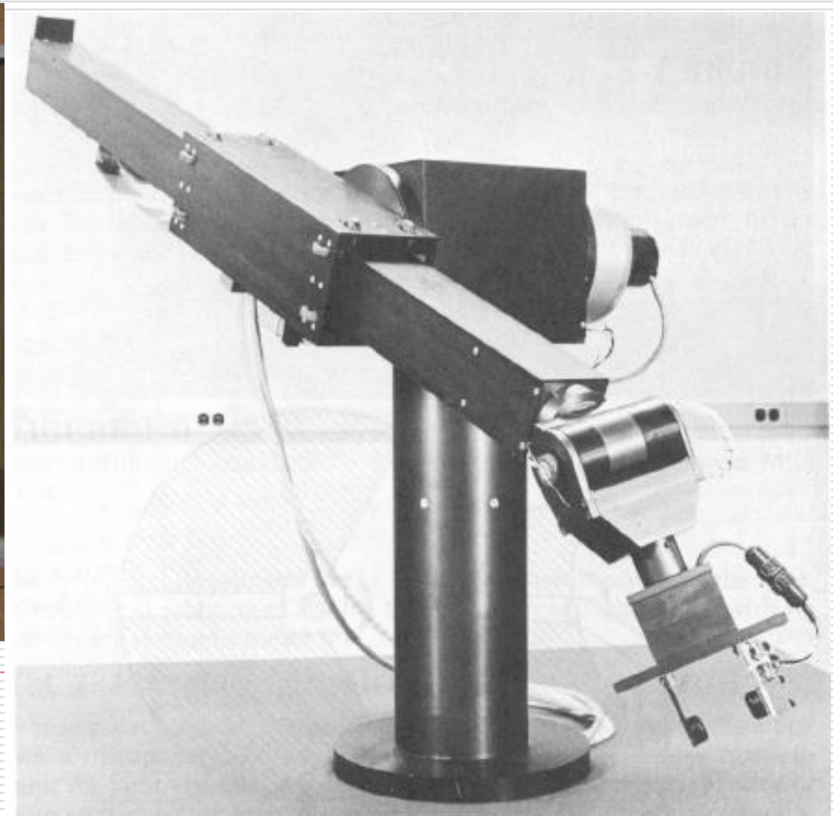
$$T_6^3 = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

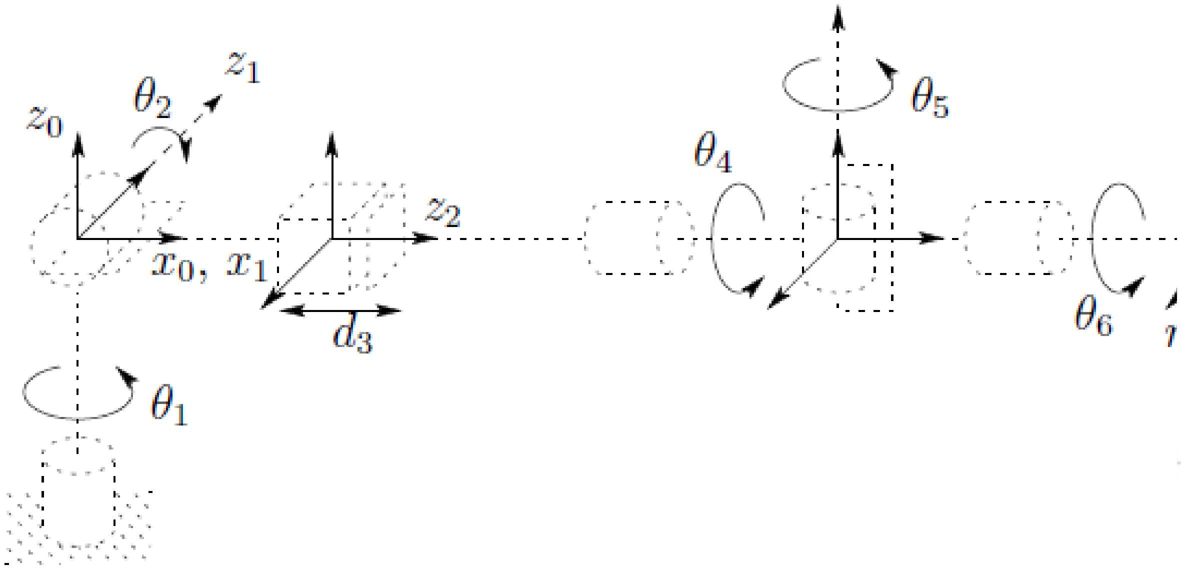
$$\begin{aligned}
r_{11} &= c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\
r_{21} &= s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\
r_{31} &= -s_4c_5c_6 - c_4s_6 \\
r_{12} &= -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\
r_{22} &= -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\
r_{32} &= s_4c_5c_6 - c_4c_6 \\
r_{13} &= c_1c_4s_5 - s_1c_5 \\
r_{23} &= s_1c_4s_5 + c_1c_5 \\
r_{33} &= -s_4s_5 \\
d_x &= c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\
d_y &= s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\
d_z &= -s_4s_5d_6 + d_1 + d_2
\end{aligned}$$

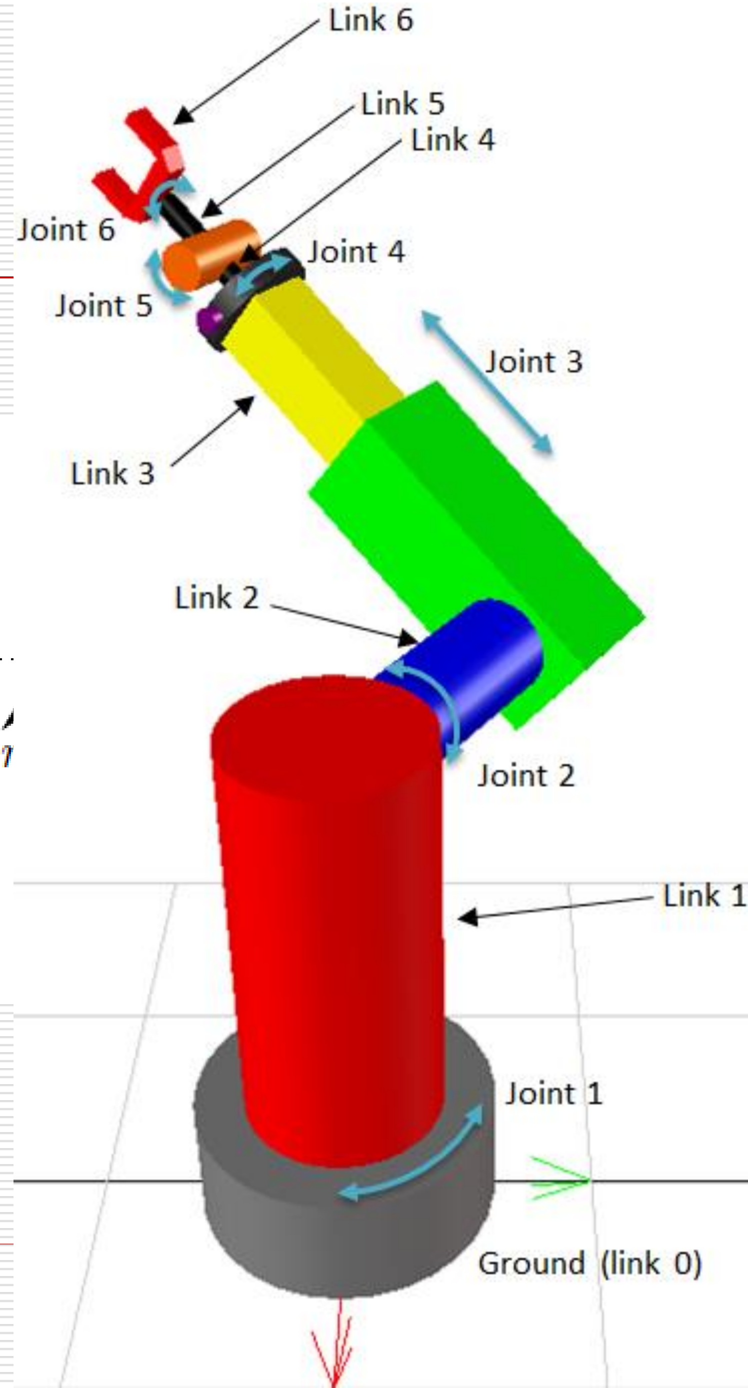
Stanford Manipulator

- This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist.



Stanford Manipulator With A Spherical Wrist





DH parameters for Stanford Manipulator

Link	d_i	a_i	α_i	θ_i
1	0	0	−90	$\theta^\star$
2	d_2	0	+90	$\theta^\star$
3	$d^\star$	0	0	0
4	0	0	−90	$\theta^\star$
5	0	0	+90	$\theta^\star$
6	d_6	0	0	$\theta^\star$

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

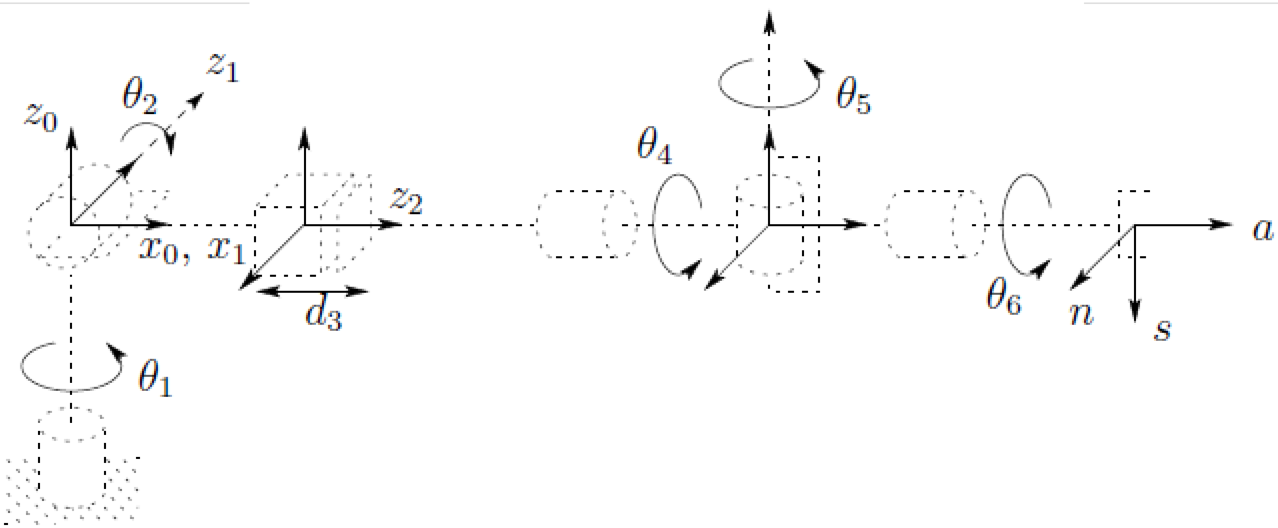
$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

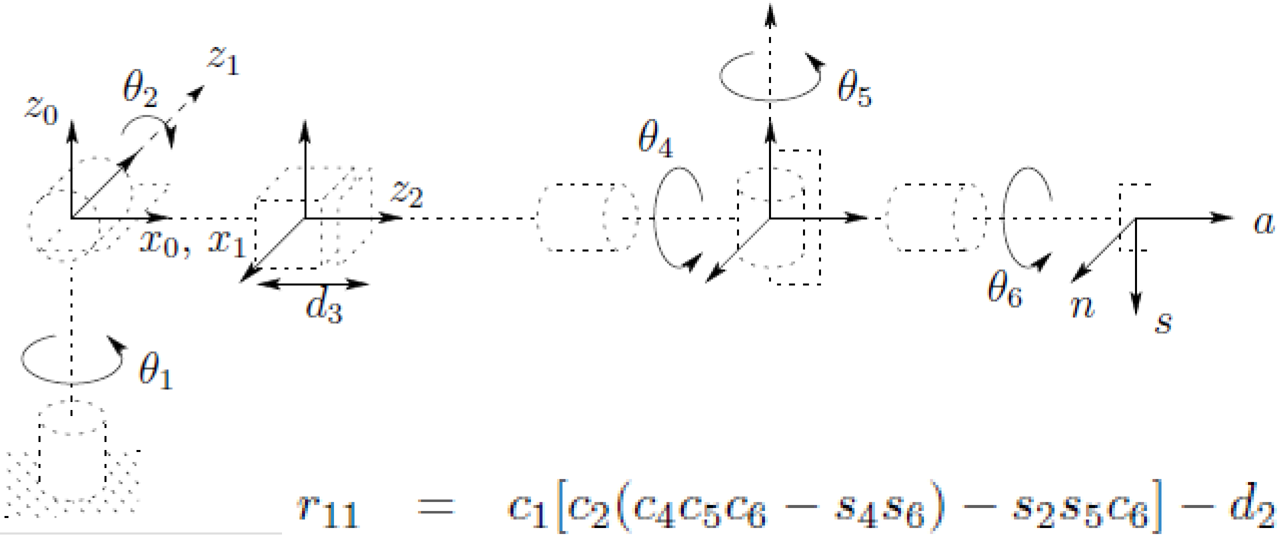
$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



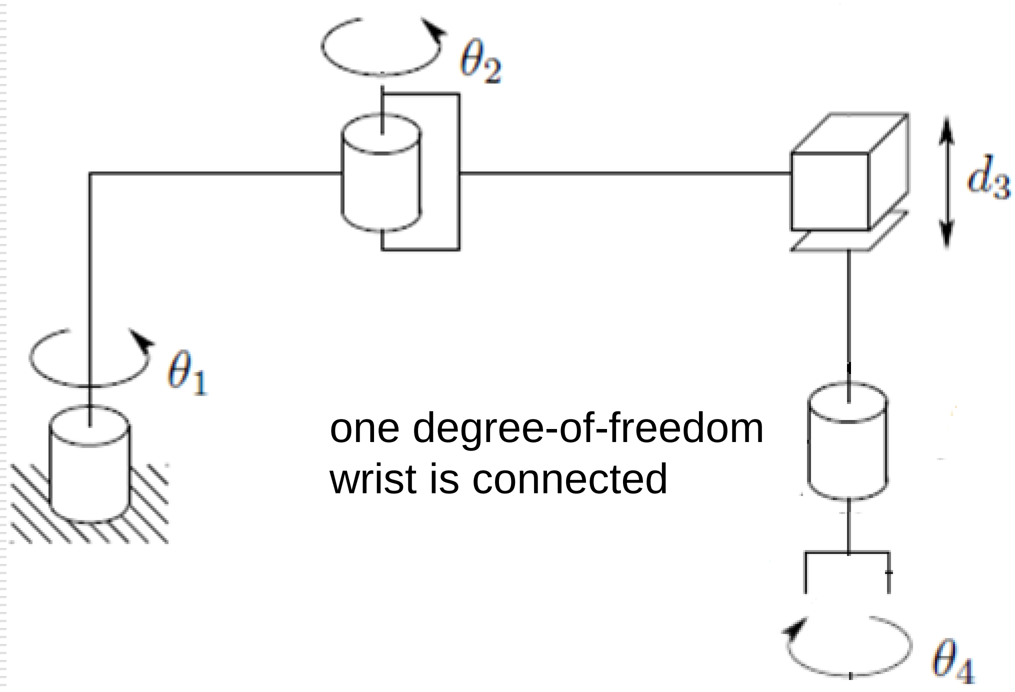


$$T_6^0 = A_1 \cdots A_6$$
$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

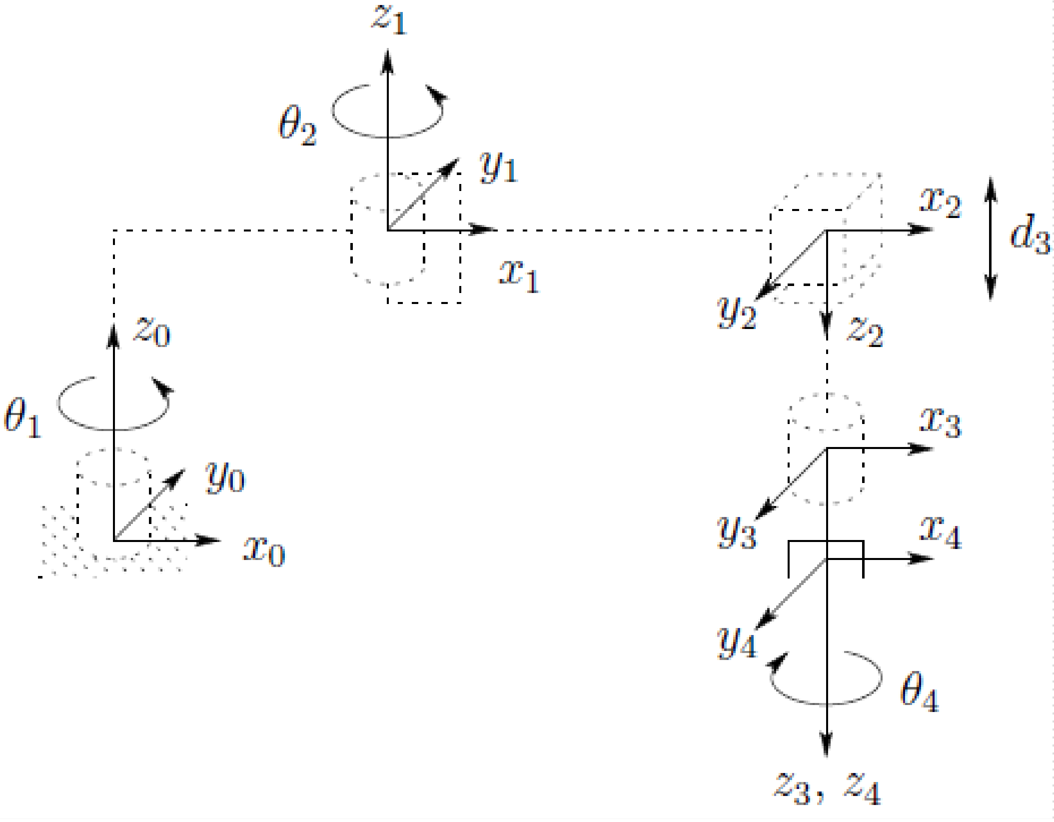
$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}$$

SCARA Manipulator

The Epson E2L653S SCARA Robot

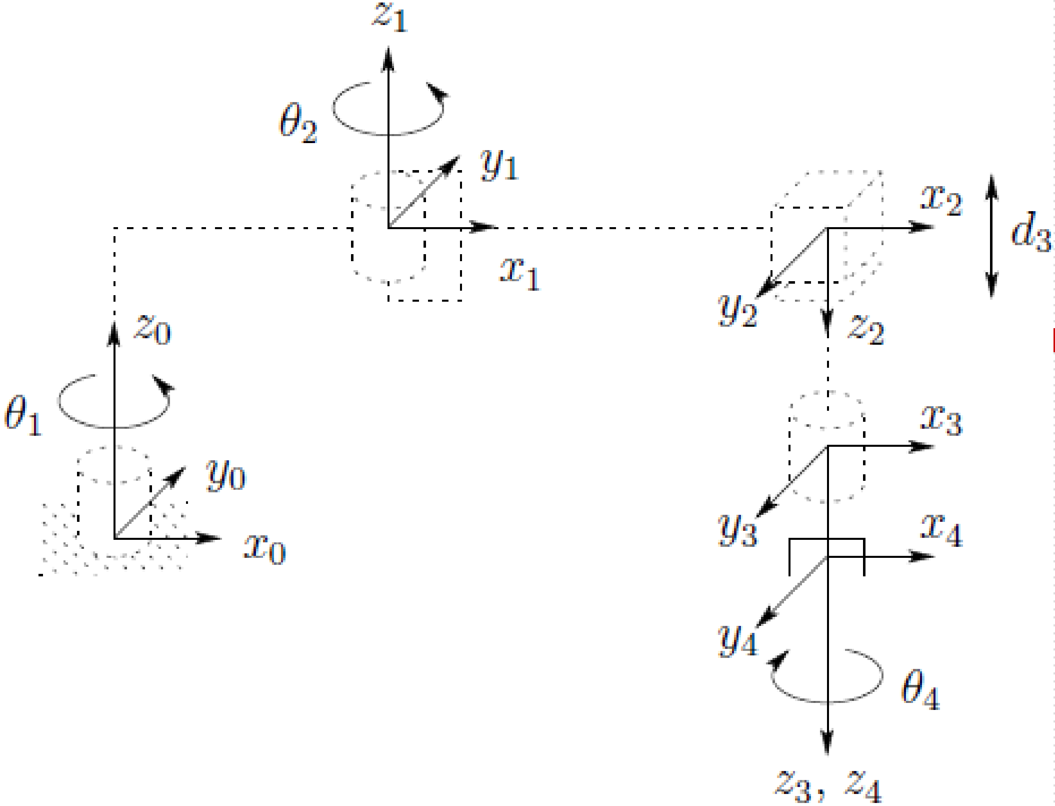


The SCARA (Selective Compliant Articulated Robot for Assembly).



DH parameters for SCARA

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

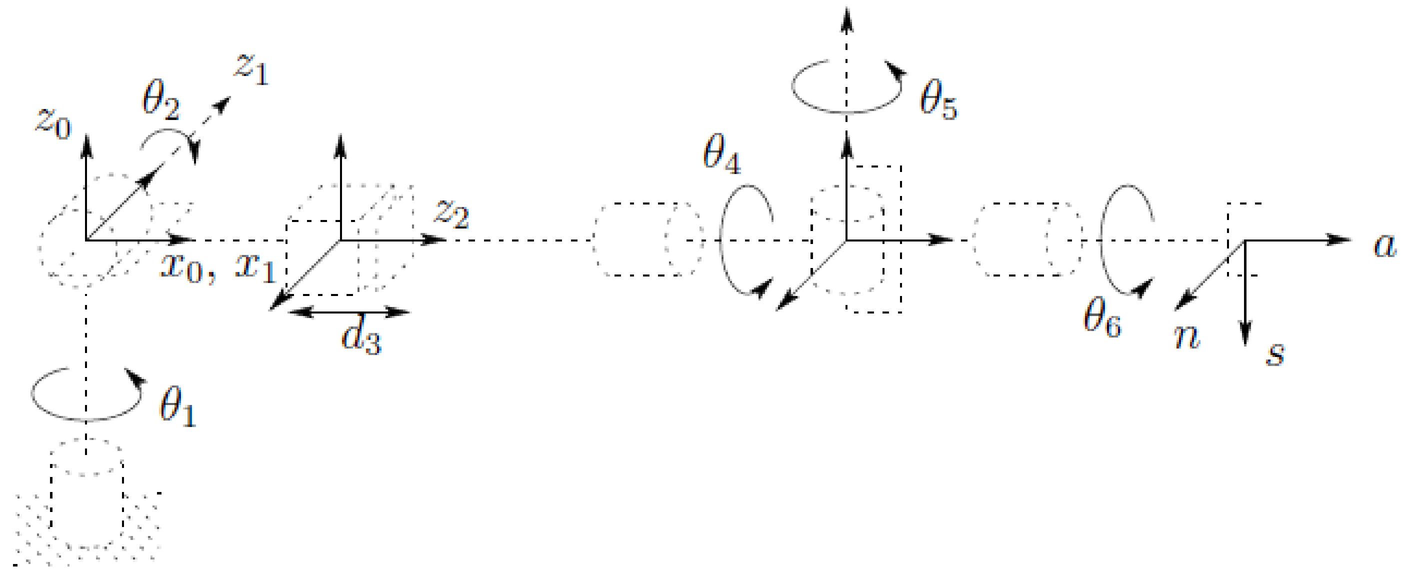
$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

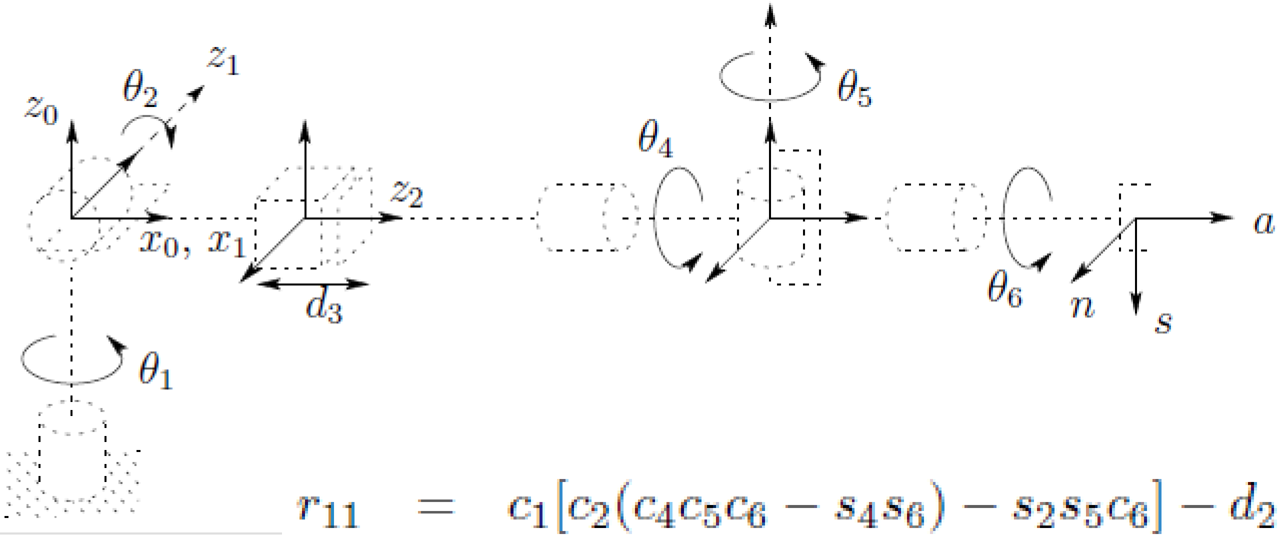
forward kinematic equations

$$\begin{aligned} T_4^0 &= A_1 \cdots A_4 \\ &= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

INVERSE KINEMATICS

Stanford Manipulator With A Spherical Wrist





$$
\begin{aligned}
T_6^0 &= A_1 \cdots A_6 \\
&= \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}
$$

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations:

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

If the values of the nonzero DH parameters are $d_2 = 0.154$ *and* $d_6 = 0.263$, *one solution to this set of equations is given by:*

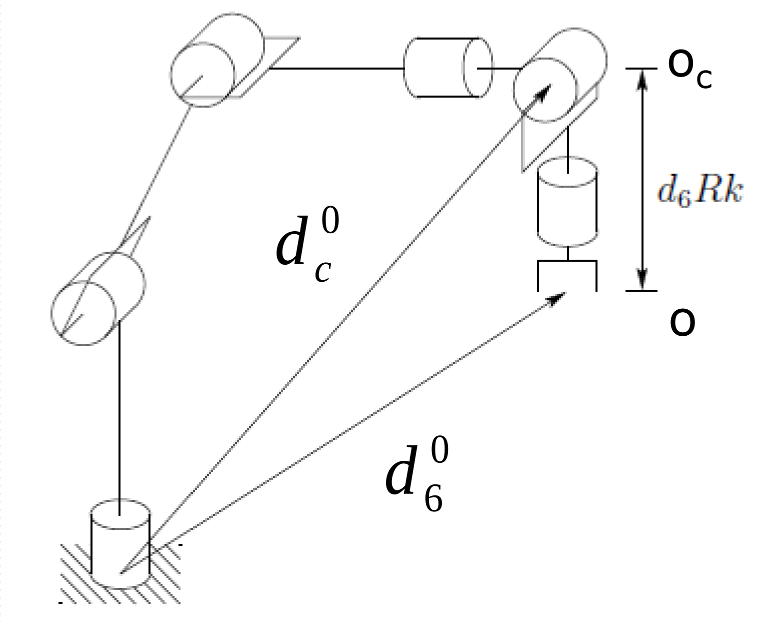
$$\theta_1 = \pi/2, \quad \theta_2 = \pi/2, \quad d_3 = 0.5, \quad \theta_4 = \pi/2, \quad \theta_5 = 0, \quad \theta_6 = \pi/2$$

DH parameters for Stanford Manipulator

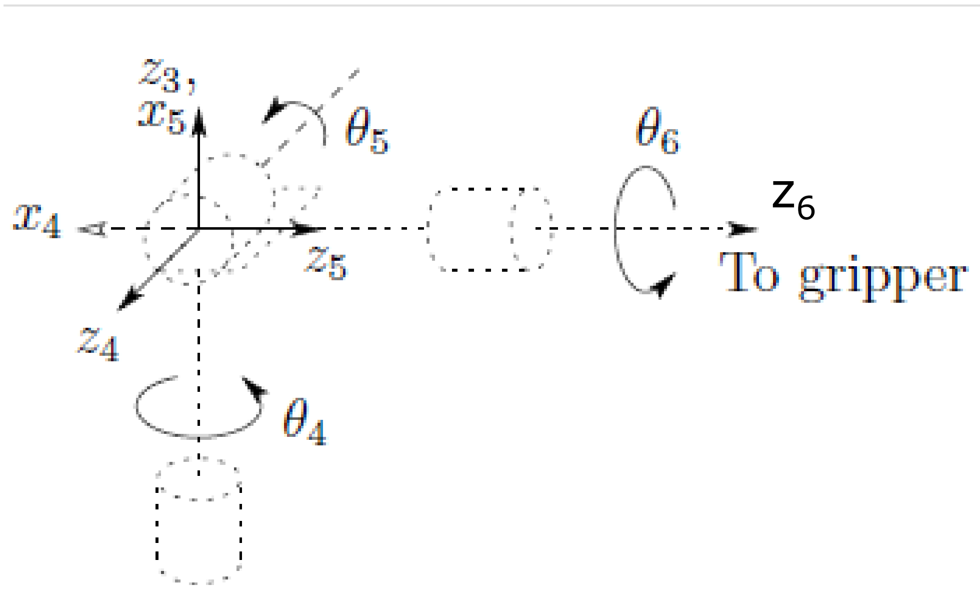
Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\theta^\star$
2	d_2	0	$+90$	$\theta^\star$
3	$d^\star$	0	0	0
4	0	0	-90	$\theta^\star$
5	0	0	$+90$	$\theta^\star$
6	d_6	0	0	$\theta^\star$

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0\\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0\\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1\\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1\\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0\\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0\\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0\\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1\\
-s_2c_4s_5 + c_2c_5 &= 0\\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154\\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763\\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

Kinematic Decoupling



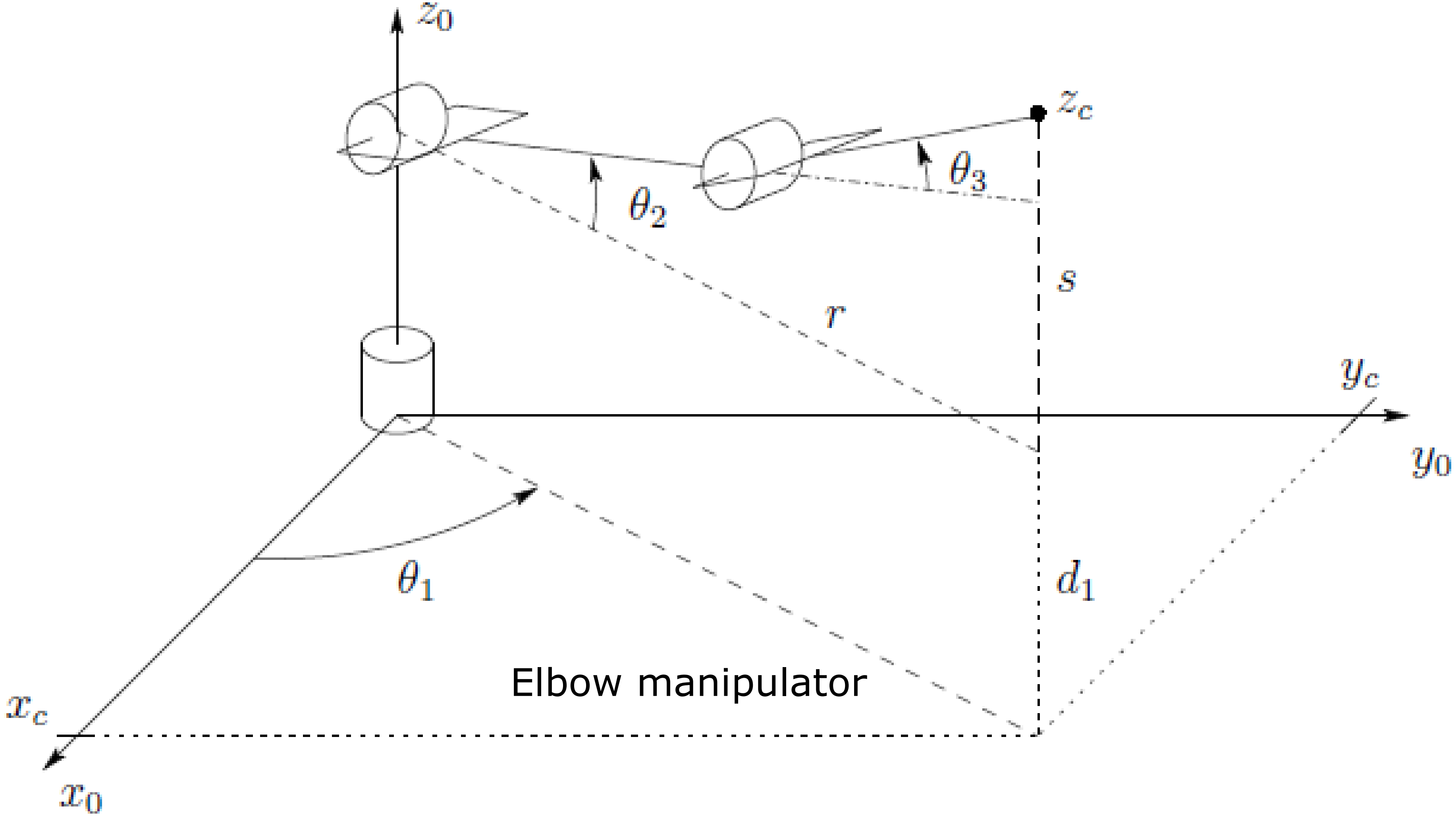
Kinematic decoupling



spherical wrist frame assignment

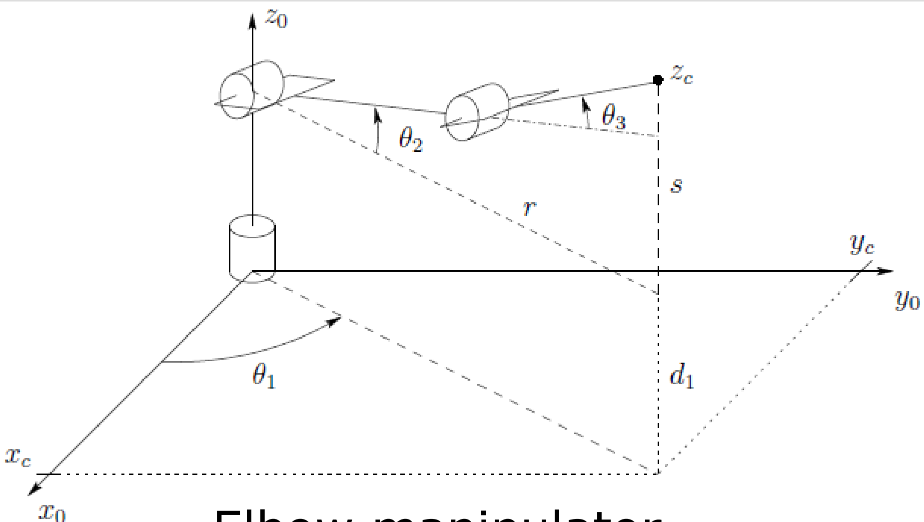
$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Inverse Position: A Geometric Approach Articulated Configuration

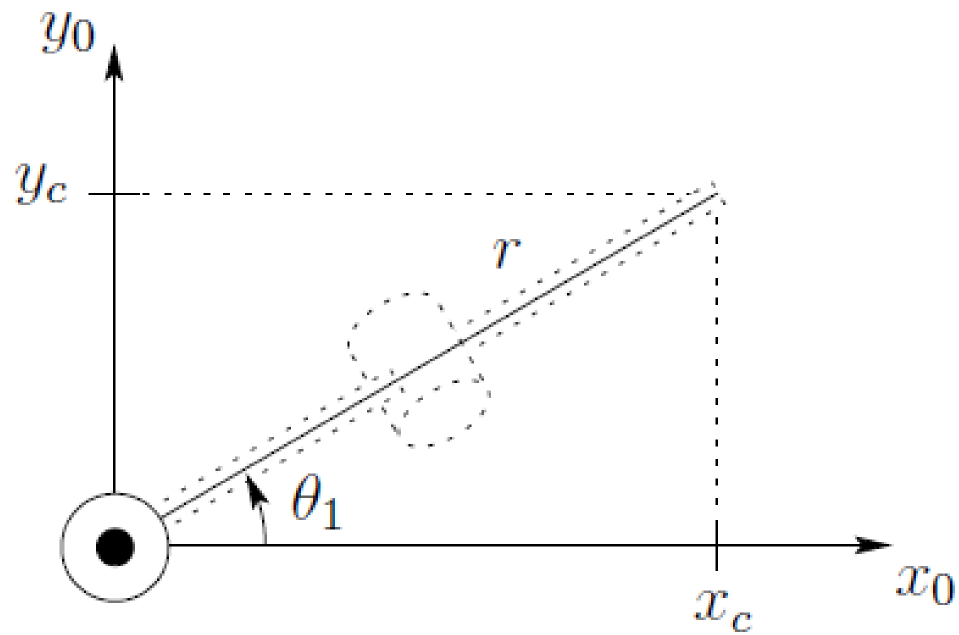


Elbow manipulator

Inverse Position: A Geometric Approach Articulated Configuration



Elbow manipulator



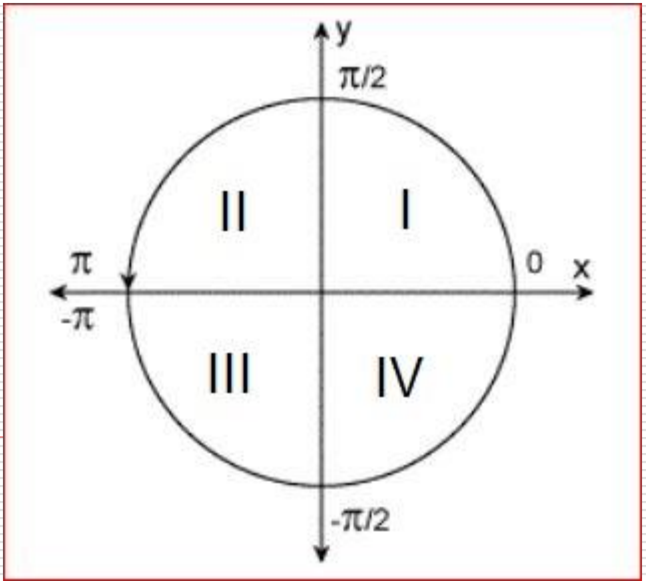
Projection of the wrist center onto x0 − y0 plane

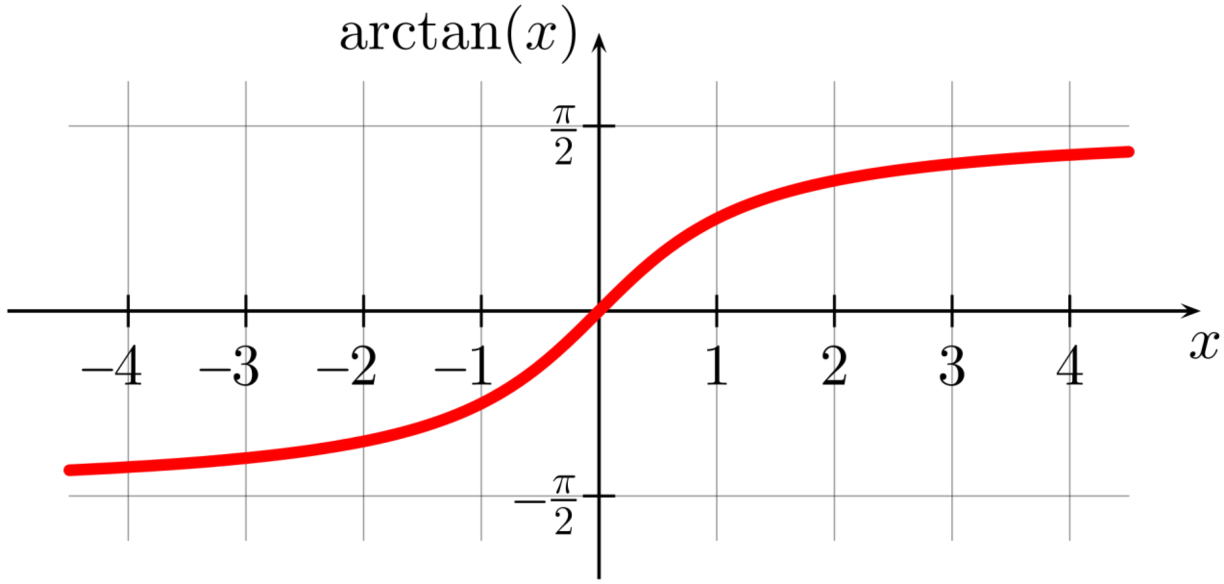
$$\theta_1 = \operatorname{atan2}(x_c, y_c) \qquad \theta_1 = \pi + \operatorname{atan2}(x_c, y_c)$$

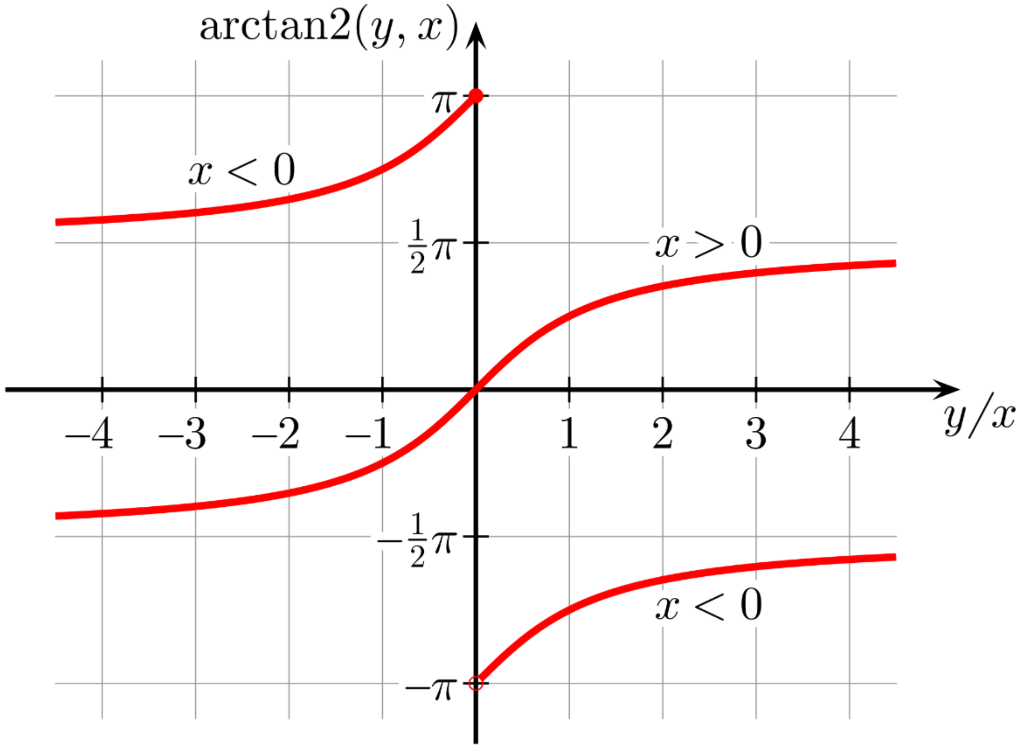
Atan vs Atan2

- Atan
  - Returns the principal value of the arc tangent of *x*, expressed in radians.
  - Notice that because of the sign ambiguity, the function cannot determine with certainty in which quadrant the angle falls only by its tangent value.
  - Principal arc tangent of *x*, in the interval [-pi/2,+pi/2] radians.
- Atan2
  - Returns the principal value of the arc tangent of y/x, expressed in radians.
  - To compute the value, the function takes into account the sign of both arguments in order to determine the quadrant.
  - Principal arc tangent of *y/x*, in the interval [-pi,+pi] radians.

$$\operatorname{atan2}(y,x)=\begin{cases}\arctan(\frac{y}{x}) & x>0\\ \arctan(\frac{y}{x})+\pi & y\geq 0\ ,\ x<0\\ \arctan(\frac{y}{x})-\pi & y<0\ ,\ x<0\\ \frac{\pi}{2} & y>0\ ,\ x=0\\ -\frac{\pi}{2} & y<0\ ,\ x=0\\ \text{undefined} & y=0\ ,\ x=0\end{cases}$$

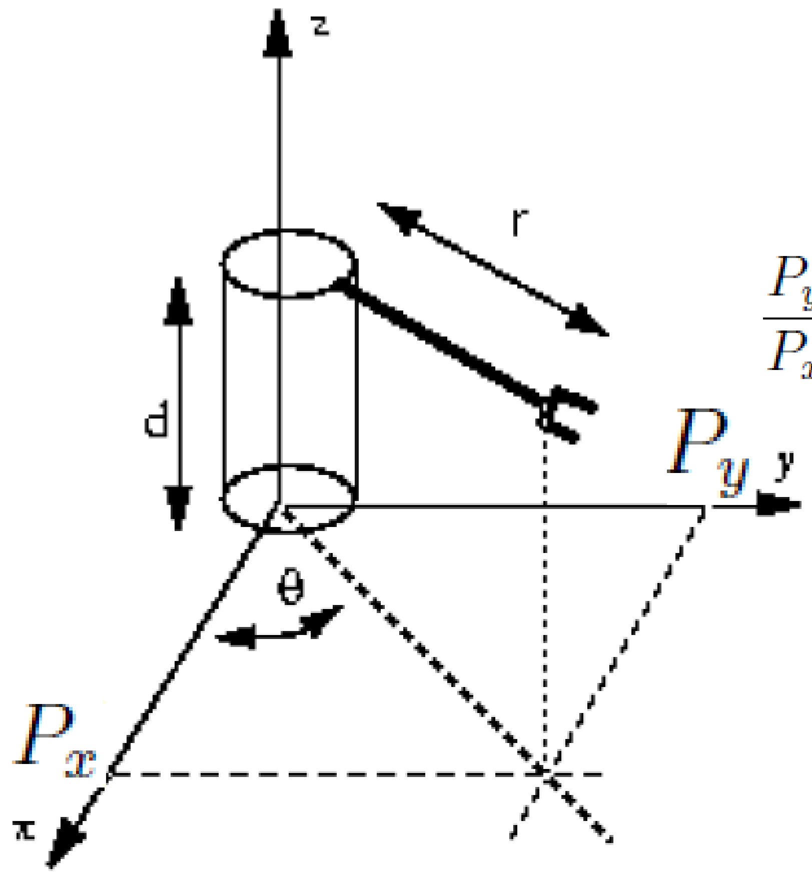



arctan(x)
π/2
−π/2
−4
−3
−2
−1
1
2
3
4
x


arctan2(y, x)
π
x < 0
½π
x > 0
−4
−3
−2
−1
1
2
3
4
y/x
−½π
x < 0
−π

Why θ1 has two solutions?

$$\theta_1 = \mathrm{atan2}(x_c, y_c) \qquad \theta_1 = \pi + \mathrm{atan2}(x_c, y_c)$$



$$P_x^2 + P_y^2 = r^2 \qquad r = \pm\sqrt{P_x^2 + P_y^2}$$

If r>0

$$\frac{P_y}{P_x} = \frac{rSin\Theta}{rCos\Theta} = Tan\Theta \ , \ \Theta = atan2(P_y \, , \, P_x)$$

Solution is (r, Θ)

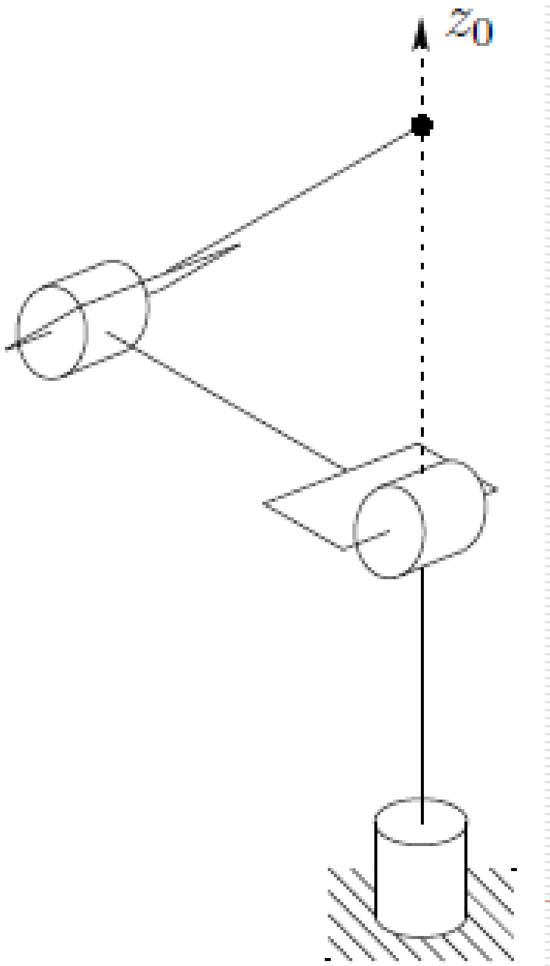
If r<0

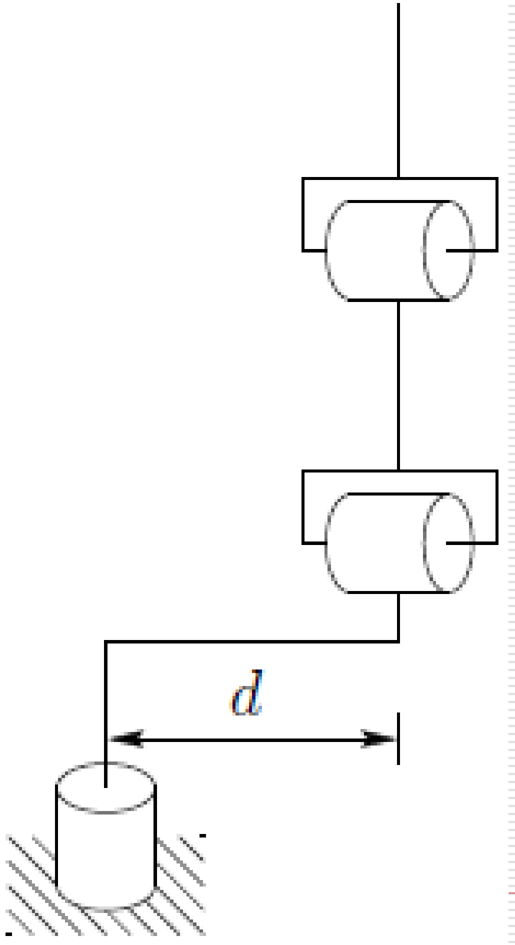
$$\Theta = atan2(-P_y \, , \, -P_x)$$

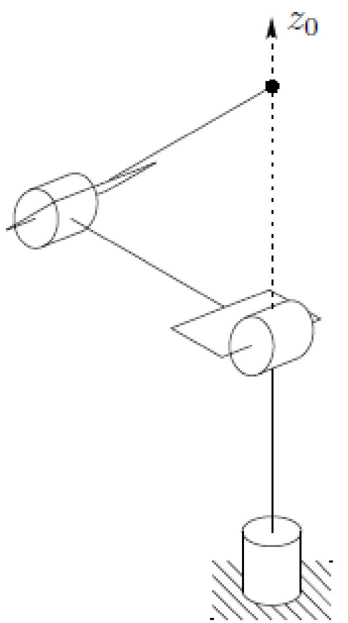
Solution is $(-r, \Theta + 180))$

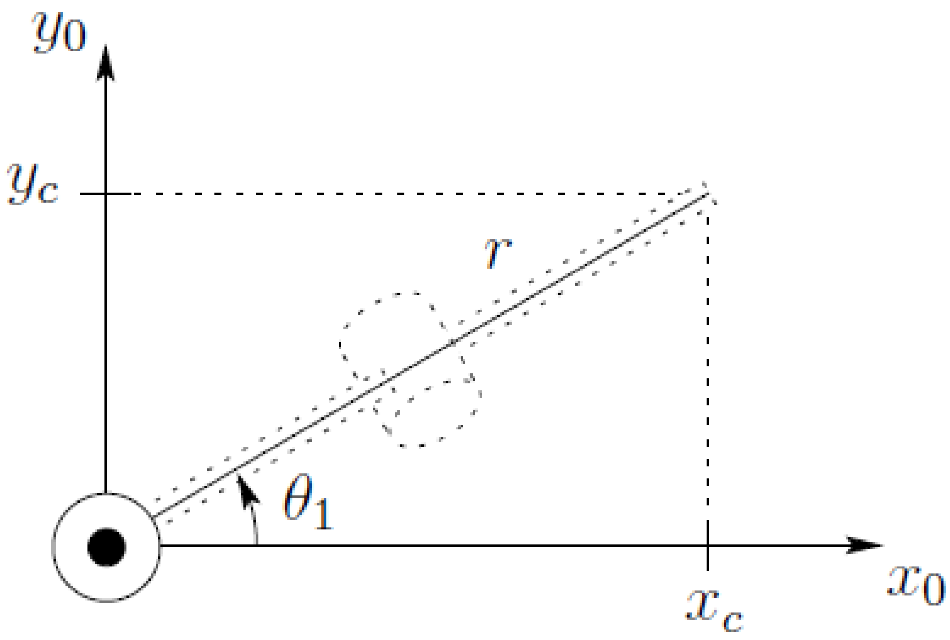
$x_c = y_c = 0$

$d \neq 0$

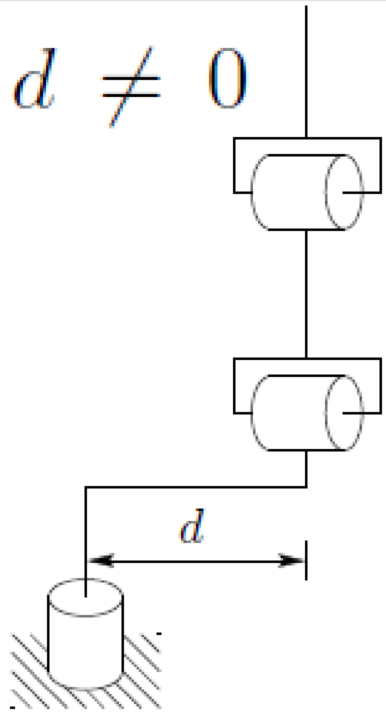




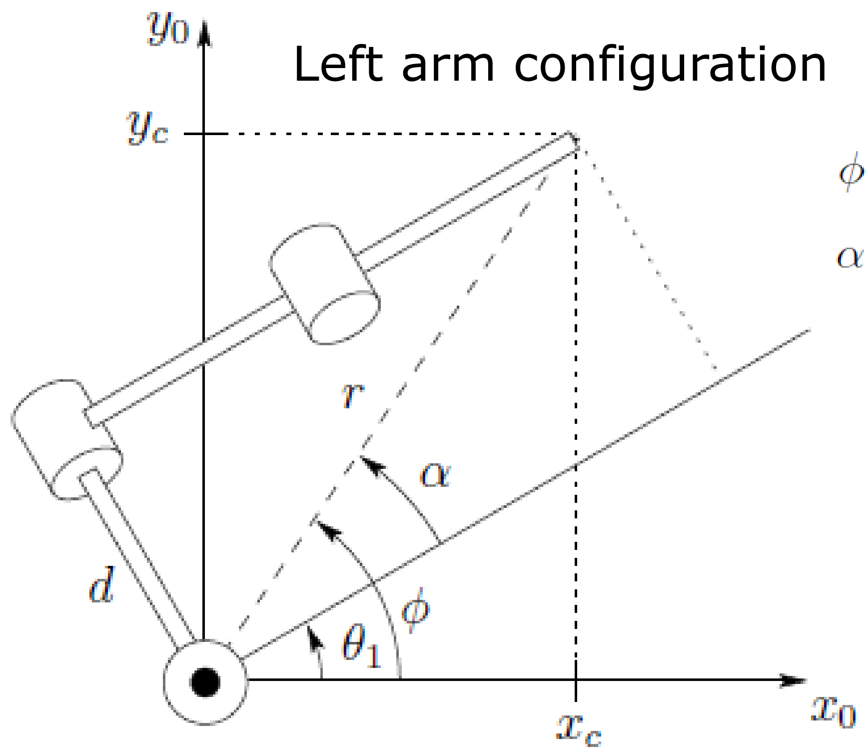




$$d \neq 0$$

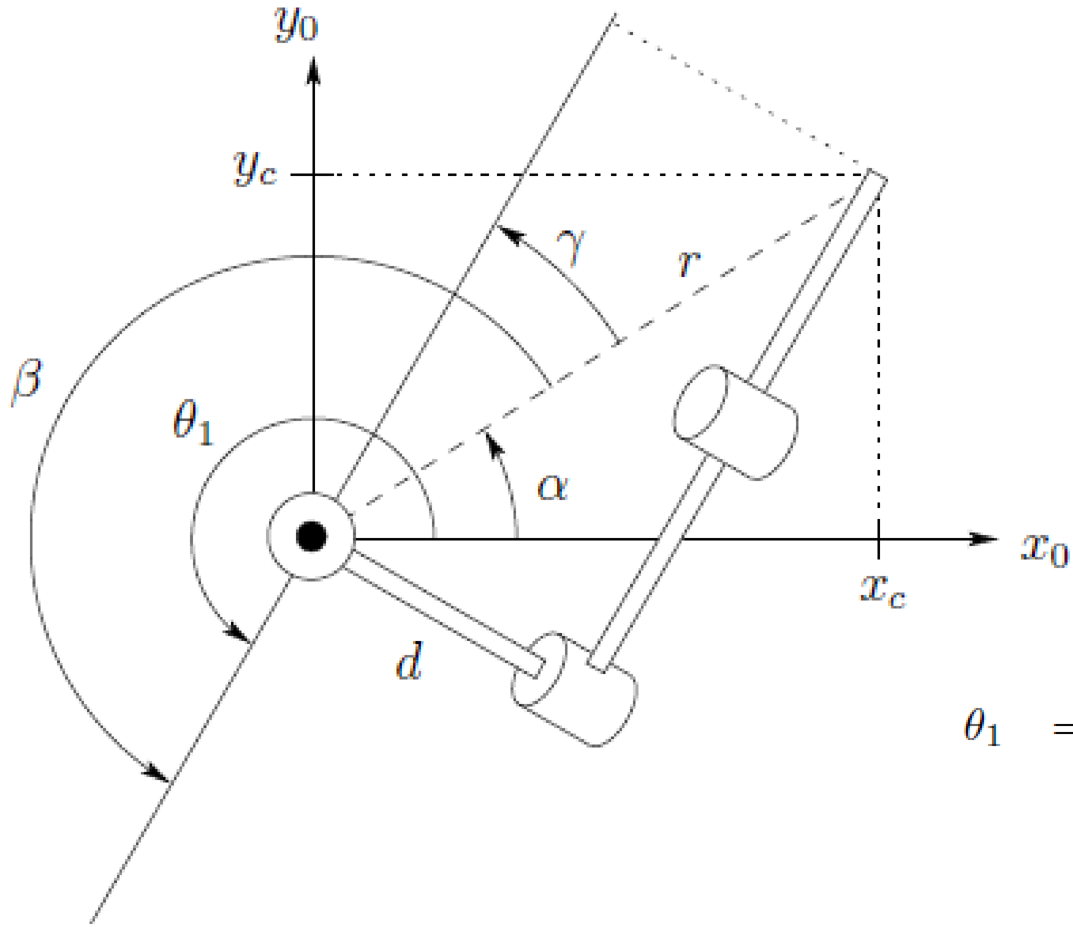


Left arm configuration



$$\theta_1 = \phi - \alpha$$

$$\begin{aligned} \phi &= \operatorname{atan2}(x_c, y_c) \\ \alpha &= \operatorname{atan2}\left(\sqrt{r^2 - d^2}, d\right) \\ &= \operatorname{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right) \end{aligned}$$



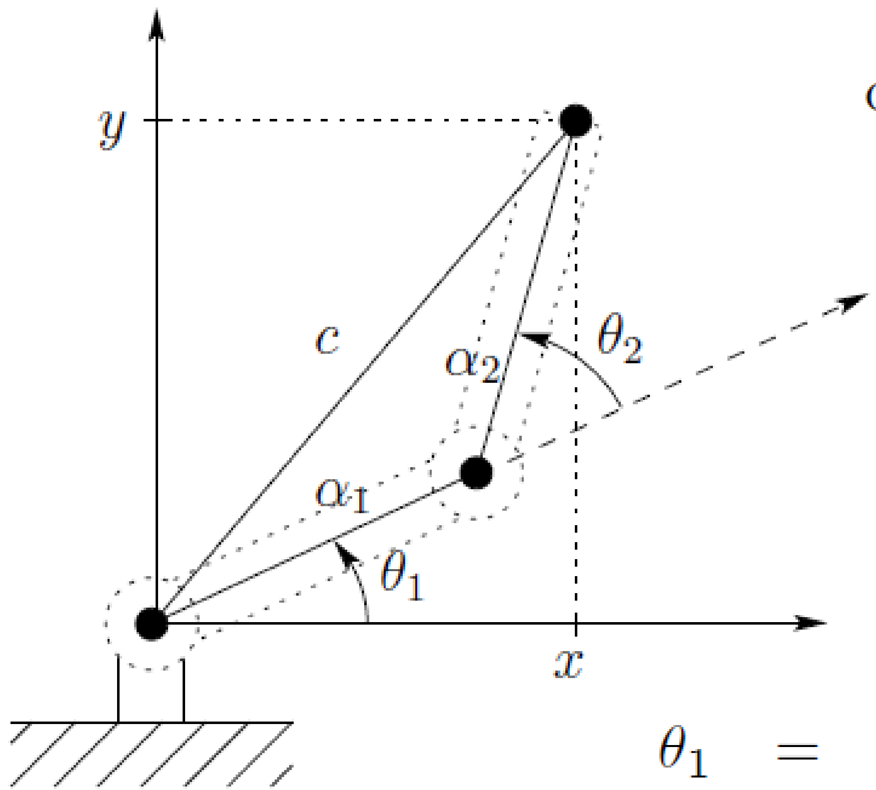
$$\theta_1 = \text{atan2}(x_c, y_c) + \text{atan2}\left(-\sqrt{r^2 - d^2}, -d\right)$$

Right arm configuration

$$\begin{aligned}
\theta_1 &= \alpha + \beta \\
\alpha &= \text{atan2}(x_c, y_c) \\
\beta &= \gamma + \pi \\
\gamma &= \text{atan2}(\sqrt{r^2 - d^2}, d)
\end{aligned}$$

$$\beta = \text{atan2}\left(-\sqrt{r^2 - d^2}, -d\right)$$

Law of Cosines



$$\cos\theta_2 = \frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2} := D$$

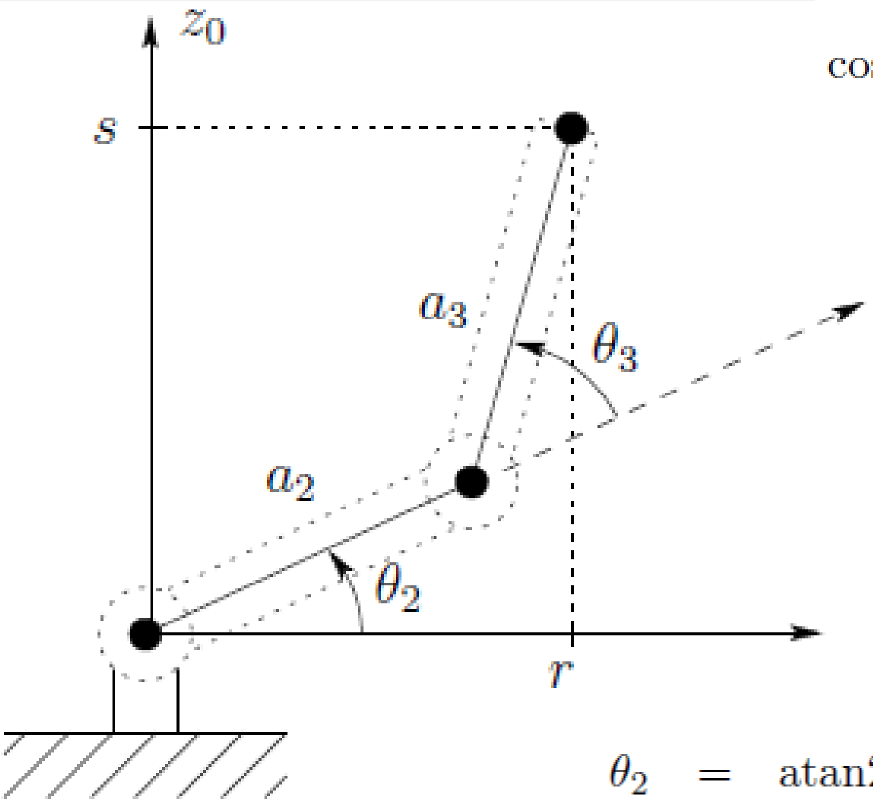
$$\theta_2 = \cos^{-1}(D)$$

$$\sin(\theta_2) = \pm\sqrt{1 - D^2}$$

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1 - D^2}}{D}$$

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2\cos\theta_2}\right)$$

Projecting onto the plane formed by links 2 and 3



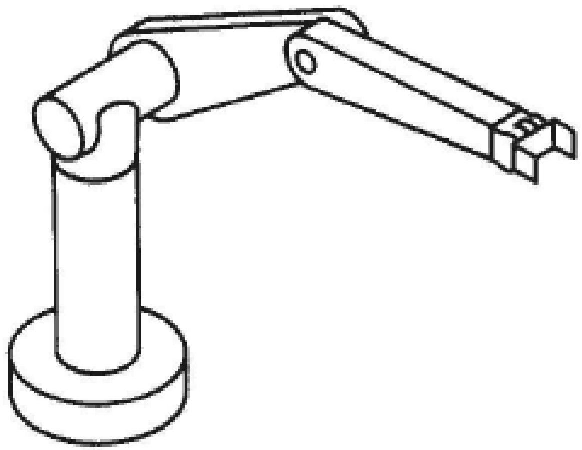
$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3} = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

$$r^2 = x_c^2 + y_c^2 - d^2 \text{ and } s = z_c - d_1$$

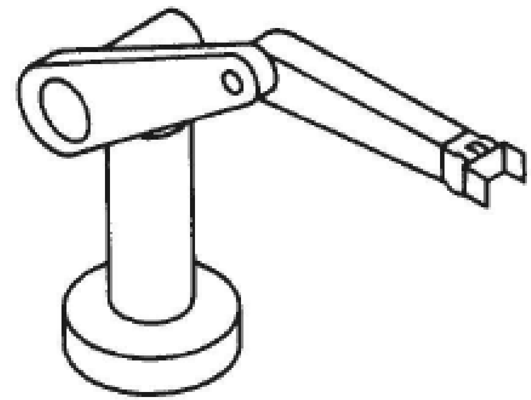
$$\theta_3 = \text{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

$$\theta_2 = \text{atan2}(r, s) - \text{atan2}(a_2 + a_3c_3, a_3s_3) = \text{atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{atan2}(a_2 + a_3c_3, a_3s_3)$$

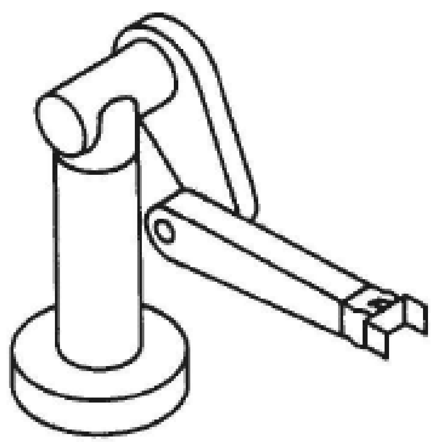
- An example of an elbow manipulator with offsets is the PUMA
- There are four solutions to the inverse position kinematics as shown.
- The two solutions for θ3 correspond to the elbow-up position and elbow-down position, respectively



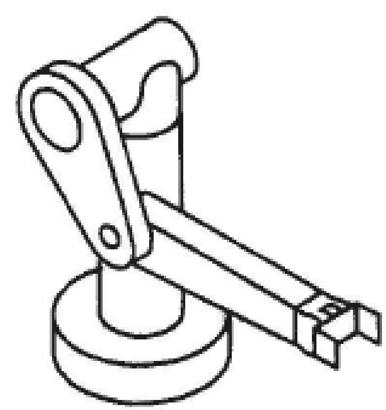
LEFT and ABOVE Arm



RIGHT and ABOVE Arm



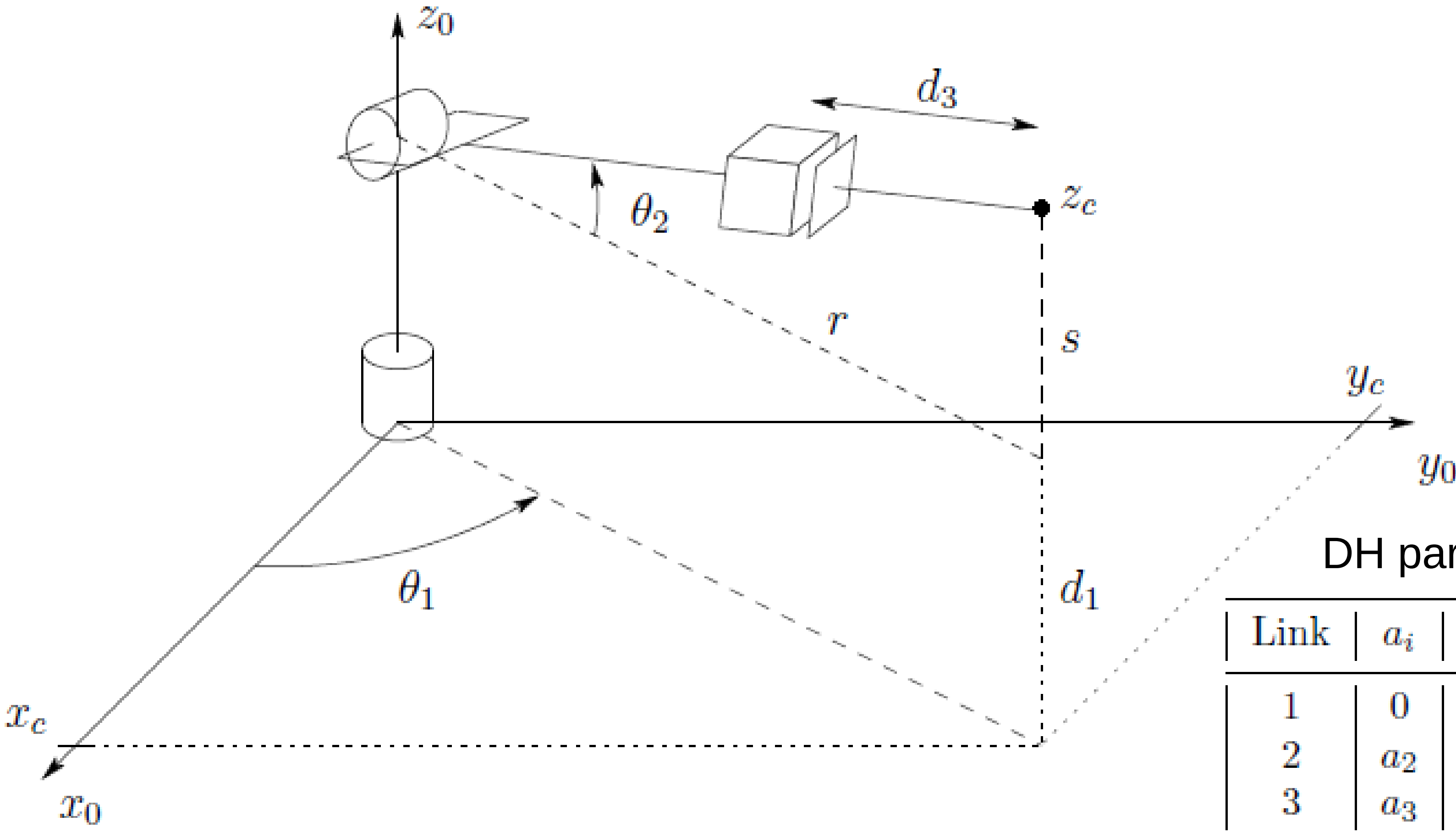
LEFT and BELOW Arm



RIGHT and BELOW Arm

Spherical Configuration

- We next solve the inverse position kinematics for a three degree of freedom spherical manipulator



$\theta_1 \quad =$

$\theta_2 \quad =$

$d_3 =$

DH parameters

Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1^*
2	a_2	0	0	θ_2^*
3	a_3	0	0	θ_3^*

Spherical Configuration

- We next solve the inverse position kinematics for a three degree of freedom spherical manipulator

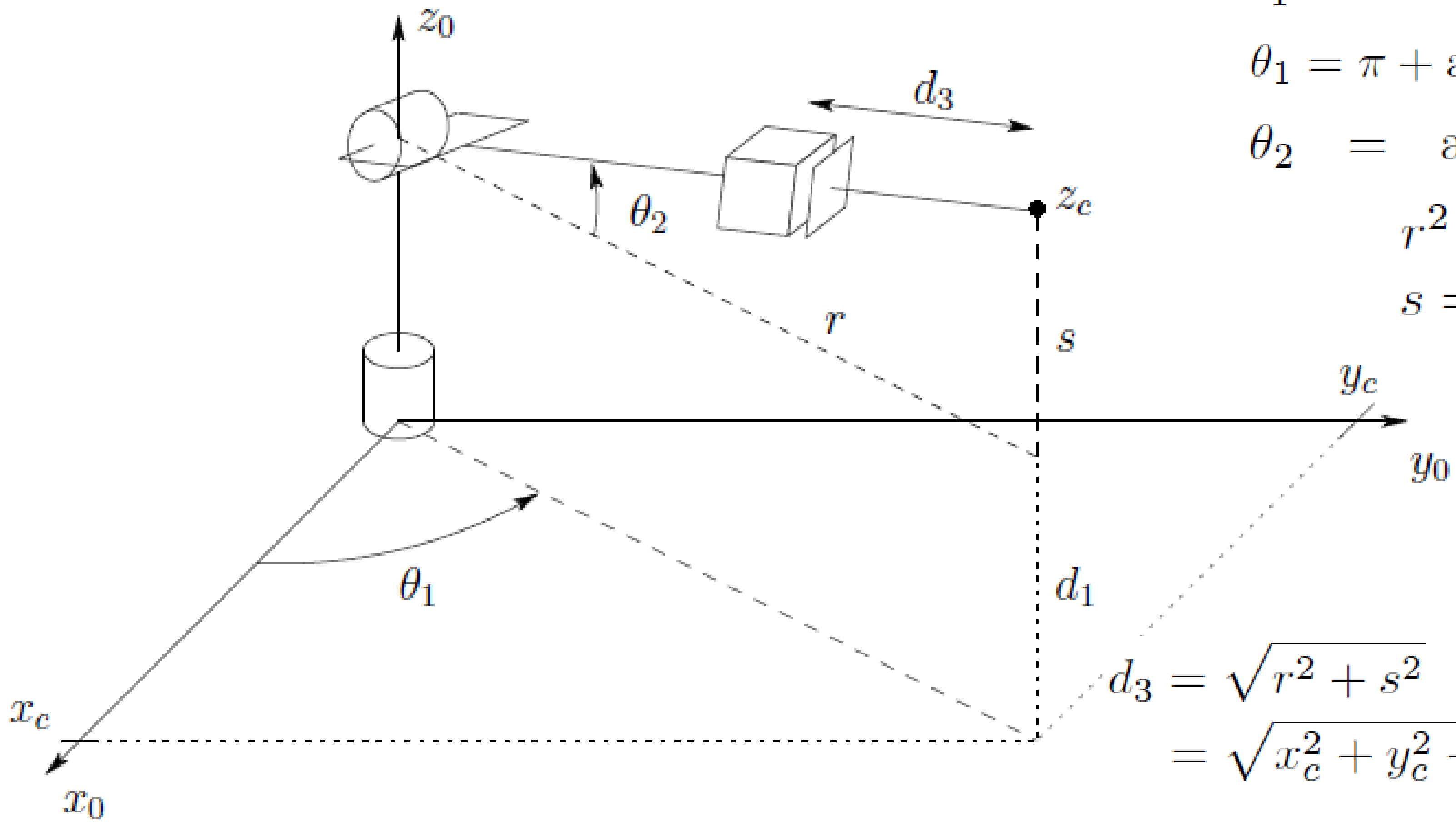
$$\theta_1 = \mathrm{atan2}(x_c, y_c)$$

$$\theta_1 = \pi + \mathrm{atan2}(x_c, y_c)$$

$$\theta_2 = \mathrm{atan2}(r, s)$$

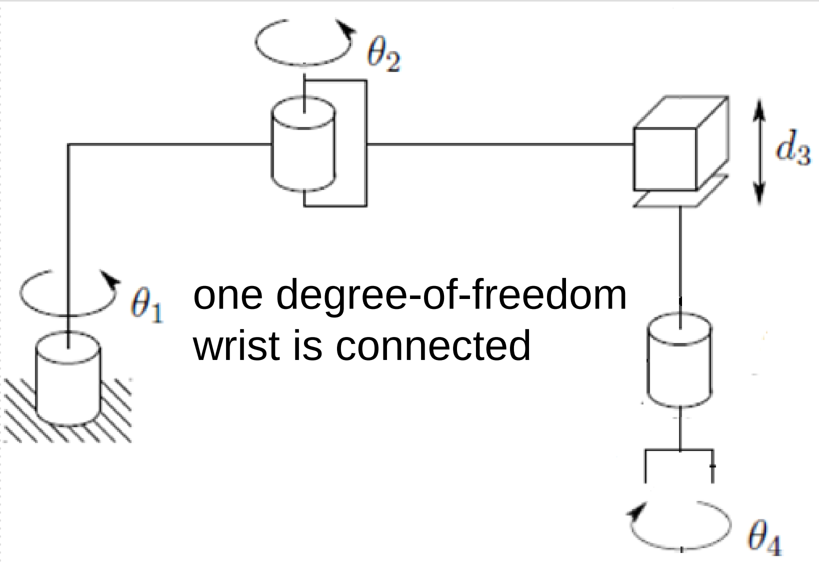
$$r^2 = x_c^2 + y_c^2$$

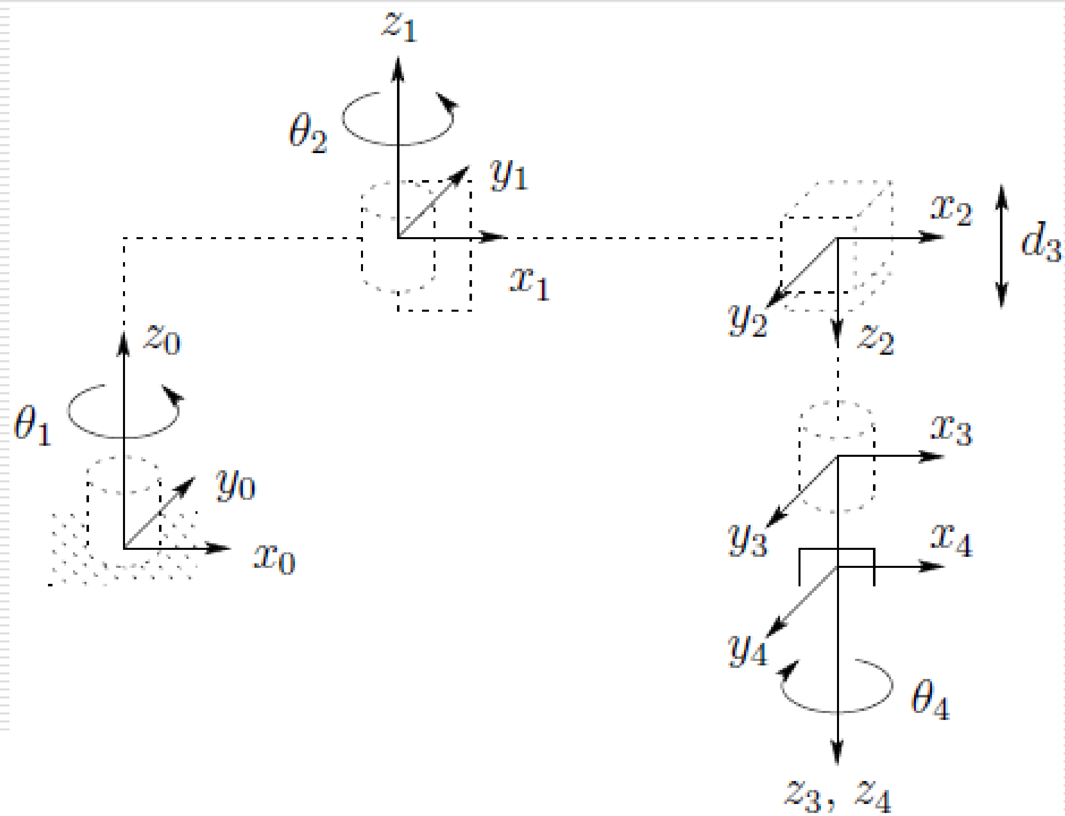
$$s = z_c - d_1$$



$$d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}$$

SCARA Manipulator





forward kinematic equations

$$
\begin{aligned}
T_4^0 &= A_1 \cdots A_4 \\
&= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

SCARA Manipulator

- We first note that, since the SCARA has only four degrees-of-freedom, not every possible H from SE(3) allows a solution.

$$
\begin{aligned}
T_4^0 &= \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

- In fact we can easily see that there is no solution unless R is of the form

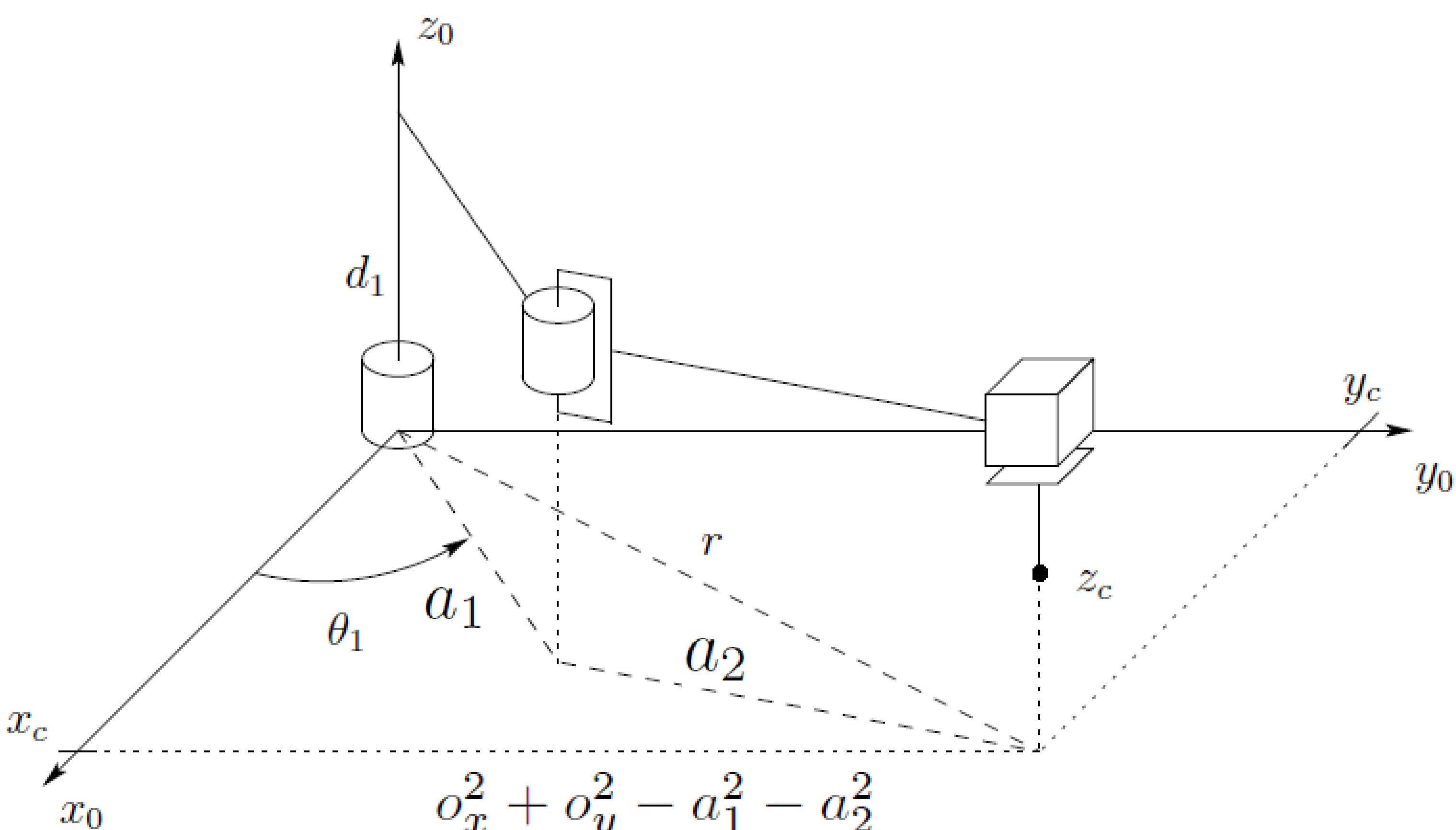
$$
R = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

- if this is the case

$$
\theta_1 + \theta_2 - \theta_4 = \alpha = \text{atan2}(r_{11}, r_{12})
$$

- Projecting the manipulator configuration onto the x0 −y0 plane immediately yields the situation of Figure. We see from this that

$$\theta_2 = \operatorname{atan2}\left(c_2, \pm\sqrt{1-c_2}\right)$$



$$c_2 = \frac{o_x^2 + o_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

$$\theta_1 = \operatorname{atan2}(o_x, o_y) - \operatorname{atan2}(a_1 + a_2c_2, a_2s_2)$$

$$\theta_4 = \theta_1 + \theta_2 - \alpha$$
$$= \theta_1 + \theta_2 - \operatorname{atan2}(r_{11}, r_{12})$$

$$d_3 = o_z + d_4$$