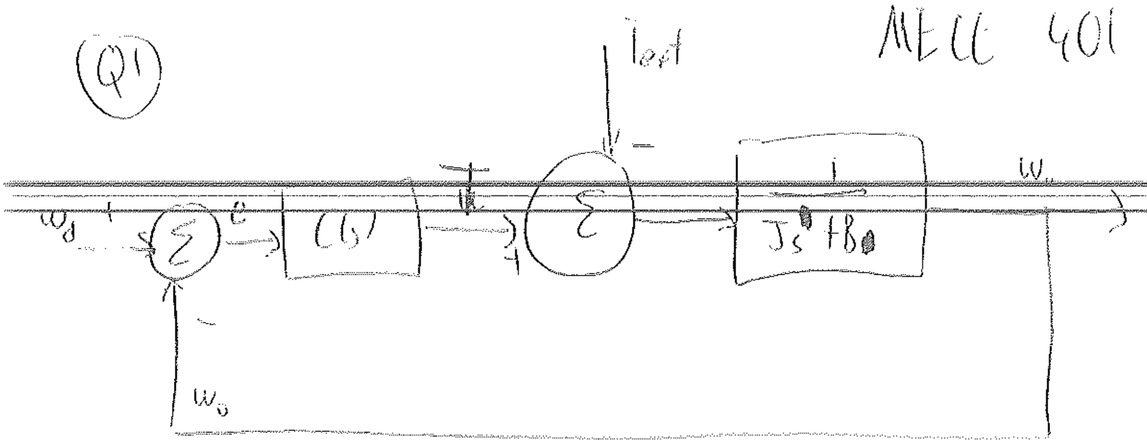
MECE 401 Final 2016-2017 Fall

Solution

(Q1)



$$w_d - w_0 = e \qquad e\,C(s) = T \qquad [T - T_{ext}]\left[\frac{1}{Js+B}\right] = w_0$$

$$\left[(w_d - w_0)\,C(s) - T_{ext}\right]\left[\frac{1}{Js+B}\right] = w_0$$

$$\left[w_d\,C(s) - w_0\,C(s) - T_{ext}\right]\left(\frac{1}{Js+B}\right) = w_0$$

$$w_d\,C(s) - T_{ext} = w_0\left[C(s) + Js + B\right] \qquad (10)$$

$$w_0 = w_d\,\frac{C(s)}{C(s)+Js+B} - T_{ext}\,\frac{1}{C(s)+Js+B} \;//$$

(a) Assume $T_{ext} = 0$ (no disturbance) and $C(s) = K_p + \frac{K_I}{s}$

$$w_0 = w_d\left[\frac{K_p + \frac{K_I}{s}}{K_p + \frac{K_I}{s} + Js + B}\right] = w_d\left[\frac{sK_p + K_I}{Js^2 + s[K_p + B] + K_I}\right]$$

$$w_0 = w_d\underbrace{\left(\frac{\frac{K_p}{J}s + \frac{K_I}{J}}{s^2 + \frac{[K_p+B]}{J}s + \frac{K_I}{J}}\right)}_{H(s)} \qquad (10)$$

(b) The system in critically damped case $\{\zeta = 1 \rightarrow$ damping ratio$\}$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + \frac{(K_p + B)}{J} + \frac{K_I}{J}$$

$$\underset{\zeta=1}{\underset{\uparrow}{2\zeta\omega_n}} = \frac{K_p + B}{J} \Rightarrow 2\omega_n = \frac{K_p + B}{J}, \quad \omega_n^2 = \frac{K_I}{J}$$

$$\frac{K_p + B}{2J} = \sqrt{\frac{K_I}{J}} \qquad K_p = 2\sqrt{JK_I} - B$$

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(c) Show that For unit step input $\omega_d = \sigma(t) \longrightarrow \omega_d(s) = \frac{1}{s}$, we have steady-state error value of $0 \frac{rad}{sec}$.

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$$e(s) = \omega_d(s) - \omega_o(s) = \omega_d - \omega_d H(s) = \omega_d \left[1 - H(s)\right] = \omega_d \left[1 - \frac{\frac{K_p}{J}s + \frac{K_I}{J}}{s^2 + \frac{(K_p + B)}{J}s + \frac{K_I}{J}}\right]$$

$$e(s) = \underset{\frac{1}{s}}{\underset{\Updownarrow}{\omega_d}} \left[\frac{s^2 + \frac{B}{J}s}{s^2 + \frac{(K_p + B)}{J}s + \frac{K_I}{J}}\right]$$

$$\lim_{s \to 0} s e(s) = \lim_{t \to \infty} e(t) = e_{ss}$$

$$= \lim_{s \to 0} s \left[\frac{s^2 + \frac{B}{J}s}{s^2 + \frac{(K_p + B)}{J}s + \frac{K_I}{J}}\right] \frac{1}{s}$$

$$= 0$$

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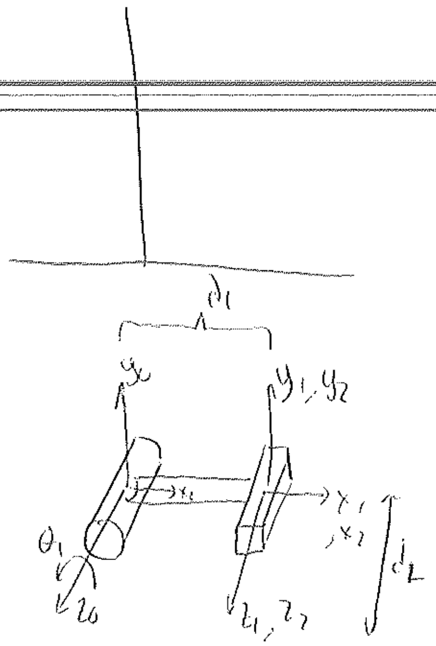
Q2



$$A_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & d_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & d_2\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

First joint revolute

Second joint prismatic

(a)

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ \\ z_0 \end{bmatrix} \qquad J_2 = \begin{bmatrix} z_1 \\ \\ 0 \end{bmatrix}$$

$$T_1 = A_1 A_2 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & d_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & d_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & d_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & d_1\sin\theta_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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(b)

$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} d_1\cos\theta_1 \\ d_1\sin\theta_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} d_1\cos\theta_1 \\ d_1\sin\theta_1 \\ d_1 \end{bmatrix}, \quad z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} d_1\cos\theta_1 - 0 \\ d_1\sin\theta_1 - 0 \\ d_1 - 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

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$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} d_1\cos\theta_1 \\ d_1\sin\theta_1 \\ d_2 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ d_1\cos\theta_1 & d_1\sin\theta_1 & d_2 \end{vmatrix} = -i\, d_1\sin\theta_1 + j\, d_1\cos\theta_1 + 0k \quad //$$

$$J = \begin{bmatrix} -d_1\sin\theta_1 & 0 \\ d_1\cos\theta_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$ 2

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{d\delta_x}{dt} \\ \frac{d\delta_y}{dt} \\ \frac{d\delta_z}{dt} \end{bmatrix} = \begin{bmatrix} -d_1\sin\theta_1 & 0 \\ d_1\cos\theta_1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{dd_2}{dt} \end{bmatrix}$$

$$\frac{dx}{dt} = -d_1\sin\theta_1 \frac{d\theta_1}{dt} \quad //$$

$$\frac{dy}{dt} = d_1\cos\theta_1 \frac{d\theta_1}{dt} \quad //$$

$$\frac{dz}{dt} = \frac{dd_2}{dt} \quad //$$

$$\frac{d\delta_z}{dt} = \frac{d\theta_1}{dt} \quad //$$

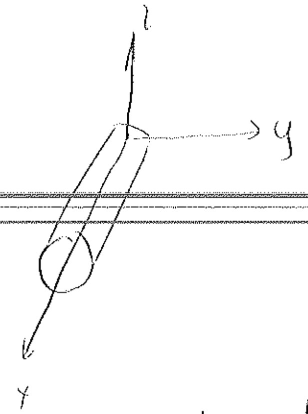
Q3 $\quad {}^{T_0}F = \begin{bmatrix} 100 \\ 50 \\ 10 \\ 0 \\ 0 \\ 50 \end{bmatrix} \qquad J = \begin{bmatrix} -0.6 & 0 \\ 0.8 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

$${}^{T_0}\tau = J^T \, {}^{T_0}f = \begin{bmatrix} -0.6 & 0.8 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 100 \\ 50 \\ 10 \\ 0 \\ 0 \\ 50 \end{bmatrix} = \begin{bmatrix} -60+40+50 \\ 10 \end{bmatrix}$$

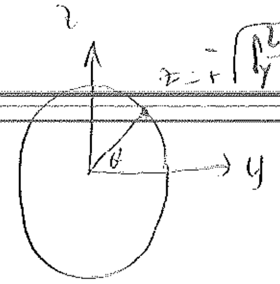
joint torques and forces

$$\begin{bmatrix} \tau_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix}$$

Q4



$x \rightarrow 0 : d$ $\qquad z^2 + y^2 = 1$



$z = \pm\sqrt{r_y^2 - y^2}$ $\qquad dm = K\, dx\, dy\, dz$

$$\boxed{\begin{aligned} y &= r\cos\theta \\ z &= r\sin\theta \\ dy\,dz &= r\,dr\,d\theta \end{aligned}}$$

$$m = \int_{x=0}^{d} \int_{y=-r_y}^{r_y} \int_{-\sqrt{r_y^2-y^2}}^{\sqrt{r_y^2-y^2}} K\, dx\, dy\, dz$$

$$m = \int_{x=0}^{d} \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} K\, r\, dr\, d\theta\, dx$$

$$m = 2\pi K \int_{x=0}^{d} \frac{r^2}{2} \Big|_0^{r_x} \qquad m = \pi r_x^2 K d$$

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$$I_{xx} = \int (y^2 + z^2)\, dm = \int_{x=0}^{d} \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} K(r^2\cos^2\theta + r^2\sin^2\theta)\, r\, dr\, d\theta\, dx$$

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$$= \int_{x=0}^{d} \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} K r^3\, dr\, d\theta\, dx = 2\pi d K \frac{r^4}{4} \Big|_0^{r_x} = 2\pi d K \frac{r_x^4}{4}$$

$$I_{yy} = \int (x^2 + z^2)\, dm = \int_{x=0}^{d} \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} K(x^2 + r^2\sin^2\theta)\, r\, dr\, d\theta\, dx$$

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$$= K \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} r \int_{x=0}^{d} (x^2 + r^2\sin^2\theta)\, d\theta\, dx\, dr$$

$$= K \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} \left[\frac{x^3}{3} r + x r^3 \sin^2\theta\right] \Big|_{x=0}^{d} d\theta$$

$$= K \int_{r=0}^{r_x} \int_{\theta=0}^{2\pi} \left[\frac{d^3}{3} r + d r^3 \sin^2\theta\right] d\theta$$

$=$

$$\cancel{= K \int_{r=0}^{r_x} \left[\frac{d^3}{3}\theta + dr\right]}$$

$$= K \int_{r=0}^{r_x}\int_{\theta=0}^{2\pi} \left[\frac{d^3 r}{3} + dr^3\left[\frac{1-\cos 2\theta}{2}\right]\right] d\theta\, dr$$

$$= K\int_{r=0}^{r_x}\left[\frac{d^3 r}{3}\theta + \frac{dr^3}{2}\theta - \frac{dr}{4}\sin 2\theta\right]\Big|_{\theta=0}^{2\pi} dr$$

$$= K\int_{r=0}^{r_x}\left[\frac{d^3}{3}(2\pi-0) + \frac{dr^2}{2}(2\pi-0) - \frac{dr}{4}\left(\sin(4\pi) - \sin(0)\right)\right] dr$$

$$= K\int_{r=0}^{r_x}\left[\frac{2d^3\pi r}{3} + \frac{dr^3}{2}2\pi\right] dr$$

$$= K\left[\frac{2d^3\pi}{3}\frac{r^2}{2} + \frac{dr^4\pi}{4}\right]\Big|_0^{r_x} = K\left[\frac{d^3\pi}{3}(r_x^2-0) + \frac{dr_x^4\pi}{4}\right]$$

$$= K\left[\frac{\pi}{3}d^3 r_x^2 + \frac{\pi}{4} dr_x^4\right]$$

$$I_{zz} = I_{yy} = K\left(\frac{\pi}{3}d^3 r_x^2 + \frac{\pi}{4}dr_x^4\right)$$

$$\int y^2 dm = \frac{I_{xx} - I_{yy} + I_{zz}}{2} = \frac{\pi d K r_x^4}{4}$$

(10)

$$\int z^2 dm = \frac{I_{xx}+I_{yy}-I_{zz}}{2} = \pi d K\frac{r_x^4}{4}$$

$$\int x^2 dm = \frac{-I_{xx}+I_{yy}+I_{zz}}{2} = \frac{-2\pi dK\frac{r_x^4}{4} + 2\left[\frac{\pi}{3}d^3 r_x^2 + \frac{\pi}{4}dr_x^4\right]}{2}$$

$$= \frac{\frac{2\pi}{3}d^3r_x^2 + \frac{2\pi}{4}dr_x^4 - 2\pi dK\frac{r_x^4}{4}}{2}$$

$x \quad y = r\cos\theta$

$$\int_{link1} xy\,dm = \int_{x=0}^{d} \int_{y=-r_y}^{r_y} \int_{-\sqrt{r_y^2}}^{\sqrt{r_y^2}} xyK\,dx\,dy\,dz$$

$$= \int_{x=0}^{d} \int_{r=0}^{r_y} \int_{\theta=0}^{2\pi} x[r\cos\theta]\,r\,dr\,d\theta\,dx$$

$$= \int_{r=0}^{r_y} \int_{\theta=0}^{2\pi} \frac{x^2}{2}\Big|_{x=0}^{d} [r^2\cos\theta]\,dr\,d\theta$$

$$= \frac{d^2}{2} \int_{\theta=0}^{2\pi} \left[\frac{r^3}{3}\Big|_0^{r_y}\right] \cos\theta\,d\theta$$

$$= \frac{d^2}{2}\frac{r_y^3}{3} \sin\theta\Big|_0^{2\pi} = 0$$

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$$\int_{link1} xz\,dm = 0$$

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$$\int_{link1} yz\,dm = \int_{x=0}^{d} \int_{r=0}^{r_y} \int_{\theta=0}^{2\pi} r^3 \sin\theta\cos\theta\,dr\,d\theta\,dx$$

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$$= 0$$

$x=$

$$\int_{link1} x\,dm = \int_{x=0}^{d} \int_{y=-r_y}^{r_y} \int_{z=-\sqrt{1-y^2}}^{+\sqrt{1-y^2}} xK\,dx\,dy\,dz$$

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$$= \int_{x=0}^{d} \int_{r=0}^{r_y} \int_{\theta=0}^{2\pi} xKr\,dx\,dr\,d\theta$$

$$= K2\pi \int_{x=0}^{d} \int_{r=0}^{r_y} xr\,dx\,dr$$

$$= K2\pi \int_{x=0}^{d} \frac{r^2}{2}\Big|_{r=0}^{r_y} x\,dx = K2\pi\frac{r_y^2}{2} \int_{x=0}^{d} x\,dx = \frac{2\pi K r_y^2}{2} \frac{x^2}{2}\Big|_0^{d}$$

$$= \frac{d^2 \pi K r_y^2}{2}$$

$$\int_{lin} y\,dm = \int_{x=0}^{d}\int_{y=-r_x}^{r_x}\int_{z=-\sqrt{r_x^2-y^2}}^{\sqrt{r_x^2-y^2}} yK\,dx\,dy\,dz \qquad y = r\cos\theta$$

$$= \int_{x=0}^{d}\int_{r=0}^{r_x}\int_{\theta=0}^{2\pi} r\cos\theta\, K\, r\,dr\,dx\,d\theta$$

$$= \int_{x=0}^{d}\int_{r=0}^{r_x}\left(\sin\theta\Big|_{\theta=0}^{2\pi}\right)\left[r^2\,dr\,dx\right] = 0$$

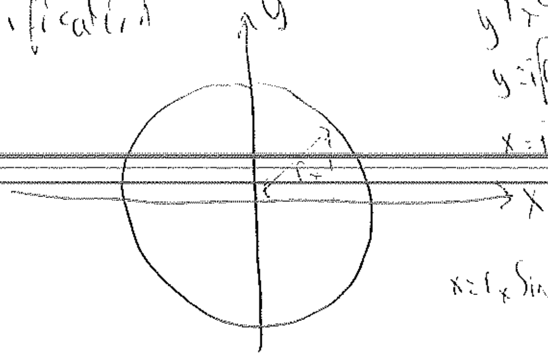
$z = r\sin\theta$

$\int z\,dm = 0$ (similarly like $\int_{lin} y\,dm$

$$J = \begin{bmatrix} \frac{2\pi}{3}d^3 r_x^2 + \frac{\pi}{2} d r_x^4 - 2\pi d K \frac{r_x^4}{4} & 0 & 0 & \frac{d^2\pi K r_x^2}{2} \\ 0 & \frac{\pi d K r_x^4}{4} & 0 & 0 \\ 0 & 0 & \frac{\pi d K r_x^4}{4} & 0 \\ \frac{d^2\pi K r_x^2}{2} & 0 & 0 & \pi r_x^2 K d \end{bmatrix}$$

cartesian to polar verification

Ek not



$y^2+x^2=r_x^2$

$y=\pm\sqrt{r_x^2-x^2}$

$x=\pm\sqrt{r_x^2-y^2}$

$x=r_x\sin\theta$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{r_x} r\,dr\,d\theta$$

$$\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\Big|_{r=0}^{r_x}\right]d\theta$$

$$=\int_{\theta=0}^{2\pi}\frac{r_x^2}{2}\,d\theta=\frac{r_x^2}{2}\left[\theta\Big|_0^{2\pi}\right]$$

$$=\pi r_x^2$$

$$\int_{y=-r_x}^{r_x}\int_{x=-\sqrt{r_x^2-y^2}}^{\sqrt{r_x^2-y^2}} dx\,dy$$

$y=r_x\cos\theta$

$$\int_{y=-r_x}^{r_x}2\left[\sqrt{r_x^2-y^2}\right]dy$$

$y=r_x\sin\theta$

$dy=r_x\cos\theta\,d\theta$

limits

$y=-r_x$

$-r_x=r_x\sin\theta$

$-1=\sin\theta \quad \theta=-\frac{\pi}{2}$

$y=r_x$

$r_x=r_x\sin\theta$

$1=\sin\theta$

$\theta=\frac{\pi}{2}$

$$\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}2\sqrt{r_x^2-r_x^2\sin^2\theta}\;r_x\cos\theta\,d\theta$$

$$\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}2\;r_x^2\sqrt{1-\sin^2\theta}\,\cos\theta\,d\theta$$

$$2r_x^2\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=2r_x^2\int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+2\cos2\theta}{2}\right)d\theta=2r_x^2\left[\frac{1}{2}\theta+\frac{\sin2\theta}{2}\right]\Bigg|_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=2r_x^2\left[\frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right]+\frac{1}{2}\left[\sin(\pi)-\sin(-\pi)\right]\right]=\pi r_y^2$$