VELOCITY KINEMATICS – THE MANIPULATOR JACOBIAN

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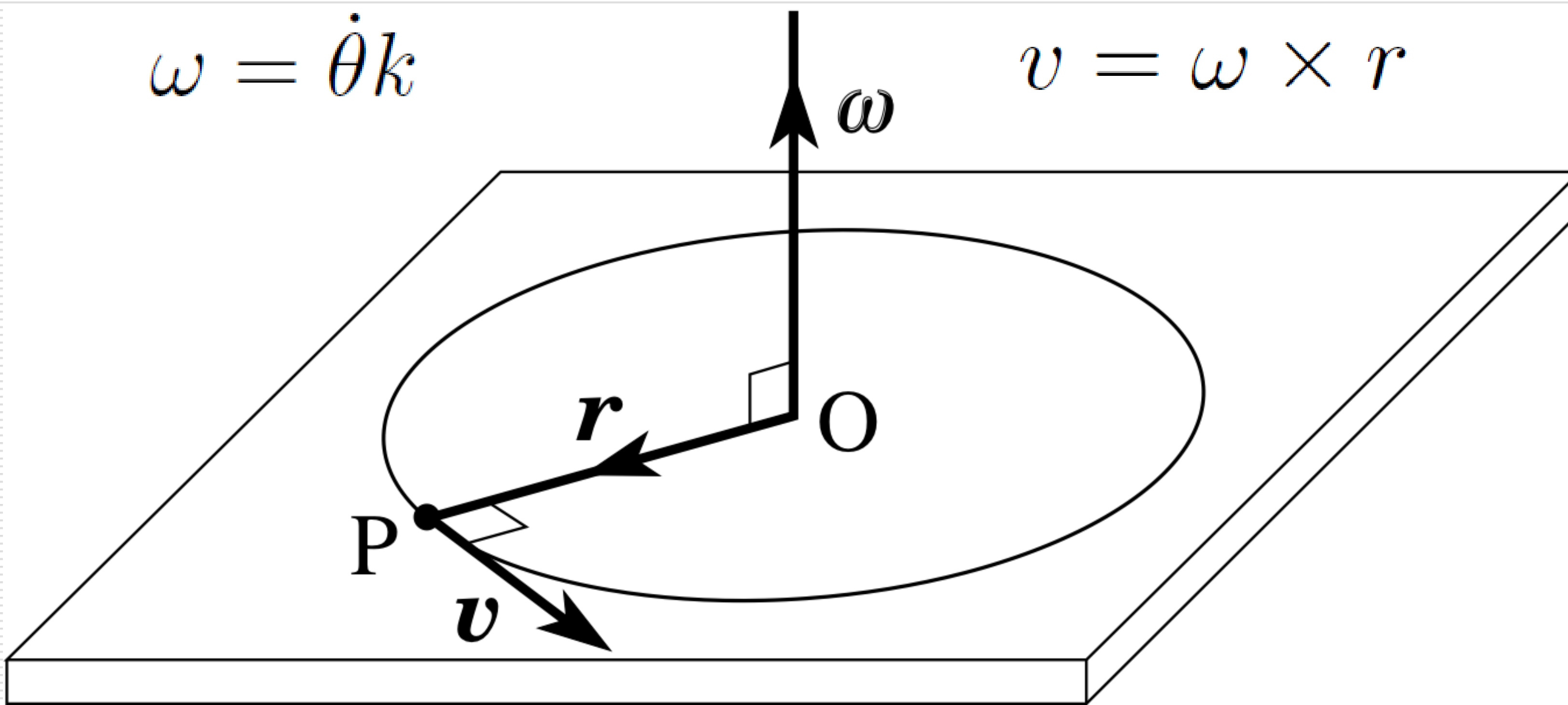
Introduction

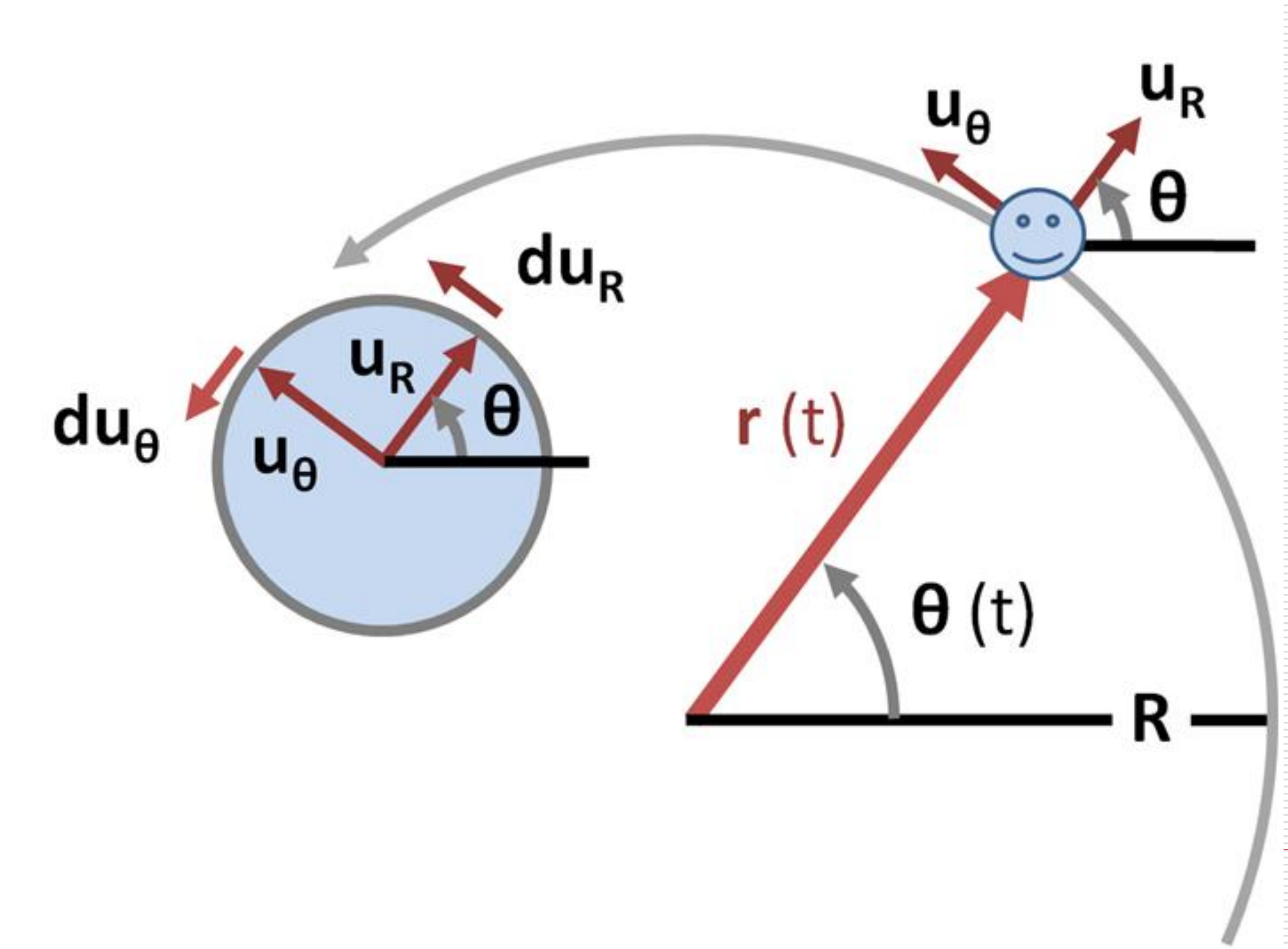
- In the previous chapter we derived the forward and inverse position equations relating
  - **joint positions** to positions and orientations of end-effector
- In this chapter we derive the velocity relationships, relating the
  - **joint velocities** to linear and angular velocities of the end-effector
- The velocity relationships are then determined by the **Jacobian** of forward kinematic equations
- The Jacobian is a matrix that can be thought of as the vector version of the ordinary derivative of a scalar function.
- The Jacobian is one of the most important quantities in the analysis and control of robot motion.

ANGULAR VELOCITY: THE FIXED AXIS CASE

- As the body rotates, a perpendicular from any point of the body to the axis sweeps out an angle θ, and this angle is the same for every point of the body.
- If k is a unit vector in the direction of the axis of rotation, then the angular velocity is given by

$$\omega = \dot{\theta} k \qquad v = \omega \times r$$





u_R
u_θ
θ
du_R
u_R
du_θ
θ
u_θ
r (t)
θ (t)
R

- In this fixed axis case, the problem of specifying angular displacements is really a planar problem, since each point traces out a circle, Therefore, it is tempting to use $\dot{\theta}$ to represent the angular velocity.
- However, as we have already seen in Chapter 2, this choice does not generalize to the three-dimensional case, either
  - when the axis of rotation is not fixed, or
  - when the angular velocity is the result of multiple rotations about distinct axes.
- Analogous to our development of rotation matrices we will need to develop skew symmetric matrix.

Jacobian

- Jacobian relates the linear and angular velocity of the end-effector to the vector of joint velocities

$$\xi = J\dot{q}$$

$$\xi = \begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} \qquad J = \begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$$

- For an n-link manipulator Jacobian is of the form

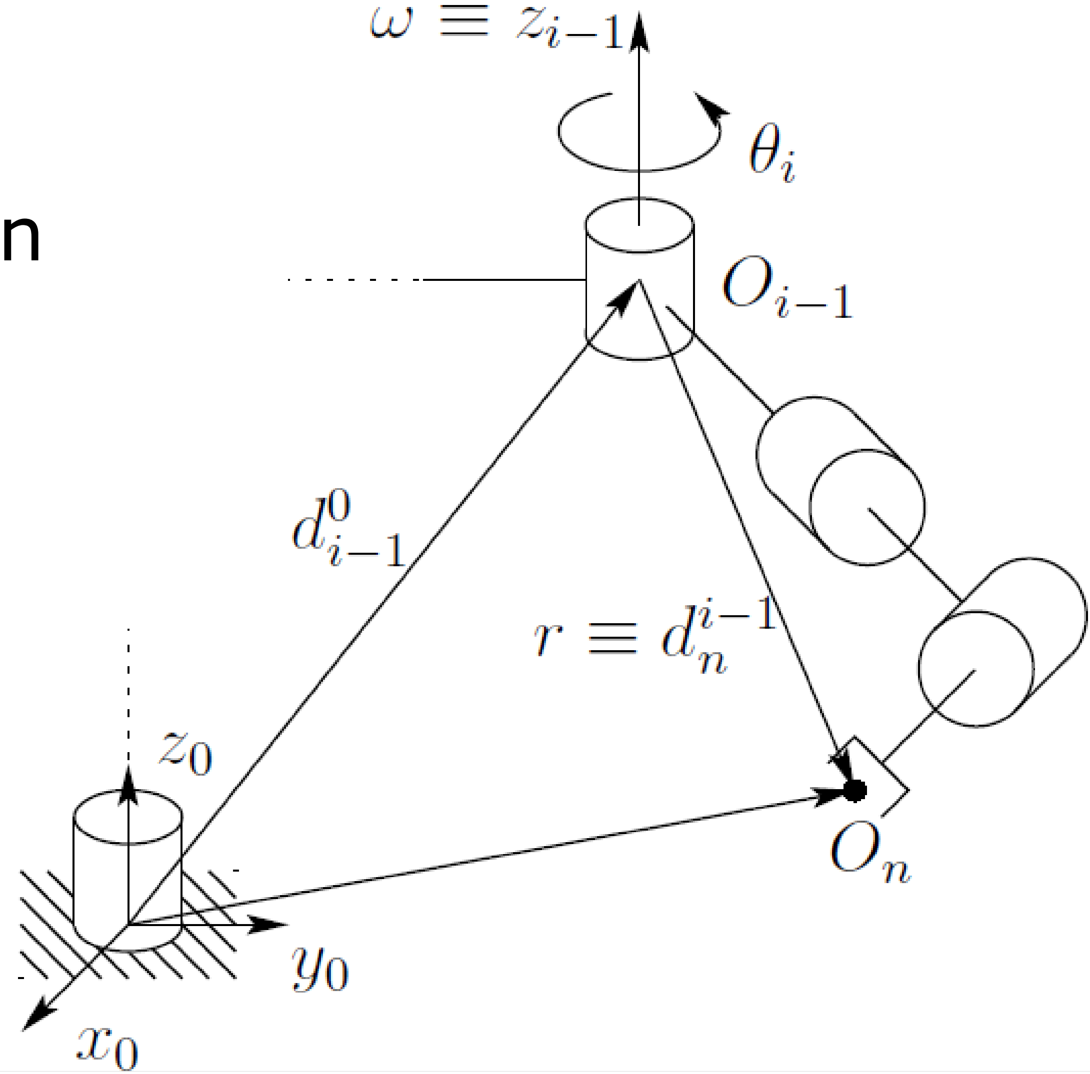
$$J = [J_1 J_2 \cdots J_n]$$

Linear part of Jacobian

- The upper half of the Jacobian $\mathbf{J_v}$ is given as

$$J_v = [J_{v_1} \cdots J_{v_n}]$$

- where the i-th column $\mathbf{J_{vi}}$ is



$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

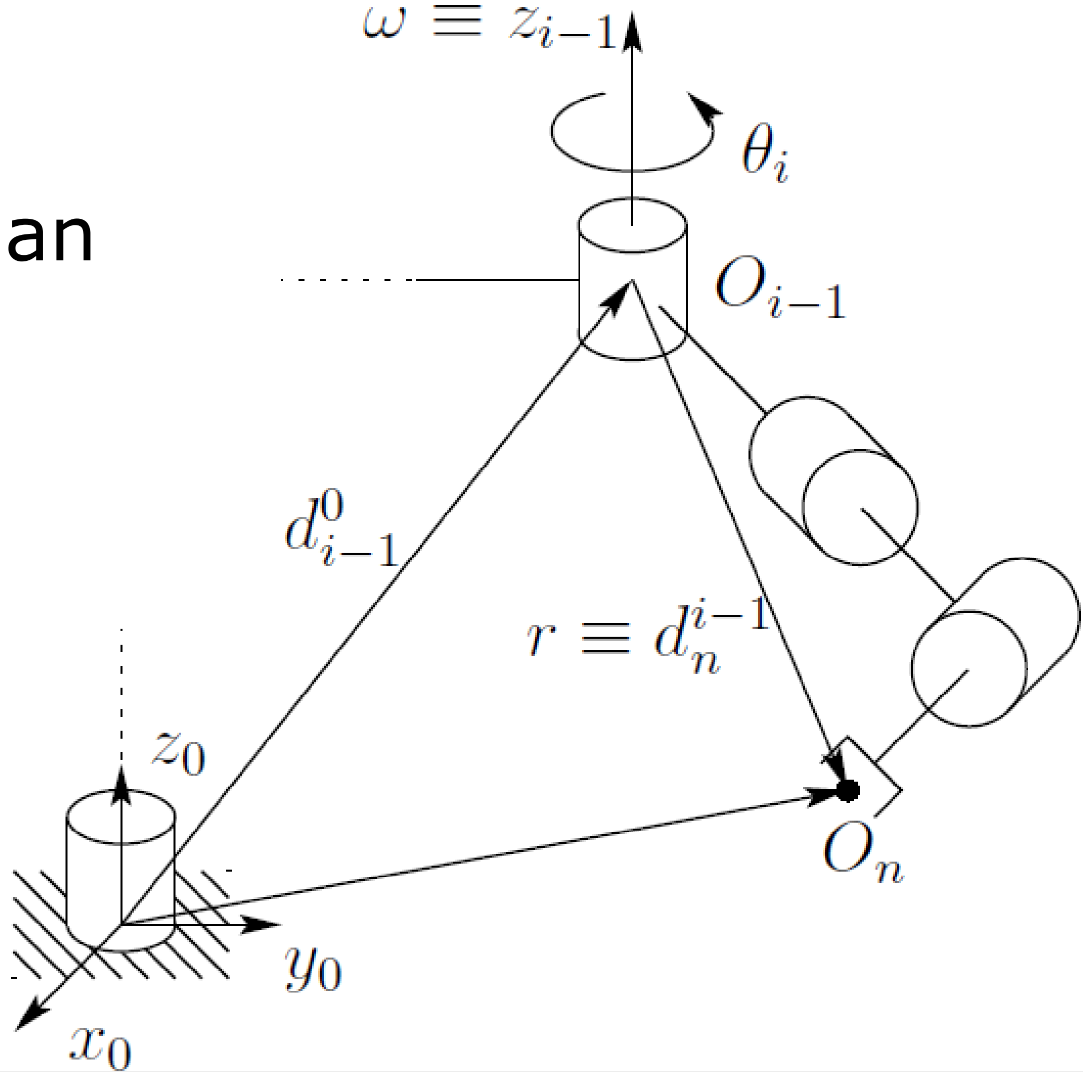
Angular part of Jacobian

- The lower half of the Jacobian $\mathbf{J_w}$ is given as

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}]$$

- where the i-th column $\mathbf{J_{wi}}$ is

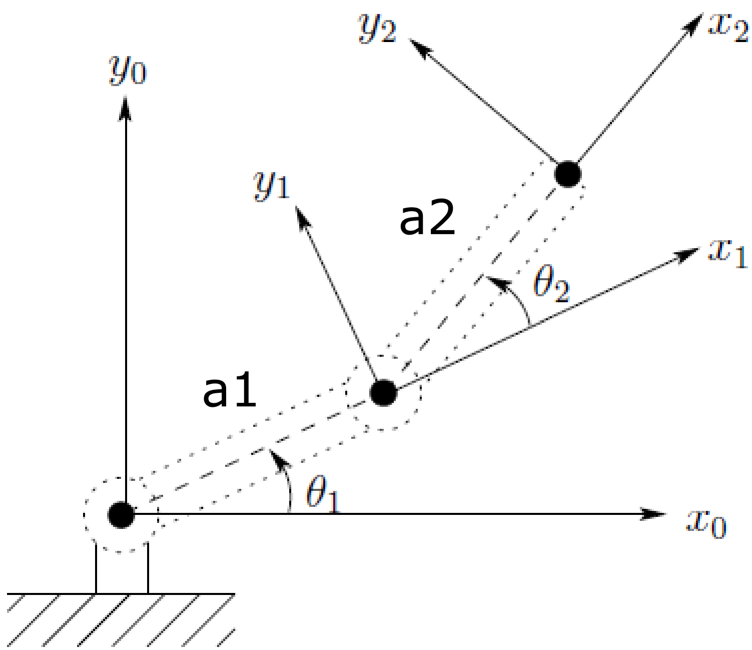
$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$



Two-Link Planar Robot Arm

- The coordinates (x, y) of the tool are

$$x = x_2 = \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2)$$
$$y = y_2 = \alpha_1 \sin\theta_1 + \alpha_2 \sin(\theta_1 + \theta_2)$$



- Differentiate equations above to obtain the relationship between the velocity of the tool and the joint velocities.

$$\dot{x} = -\alpha_1 \sin\theta_1 \cdot \dot{\theta}_1 - \alpha_2 \sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$
$$\dot{y} = \alpha_1 \cos\theta_1 \cdot \dot{\theta}_1 + \alpha_2 \cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)$$

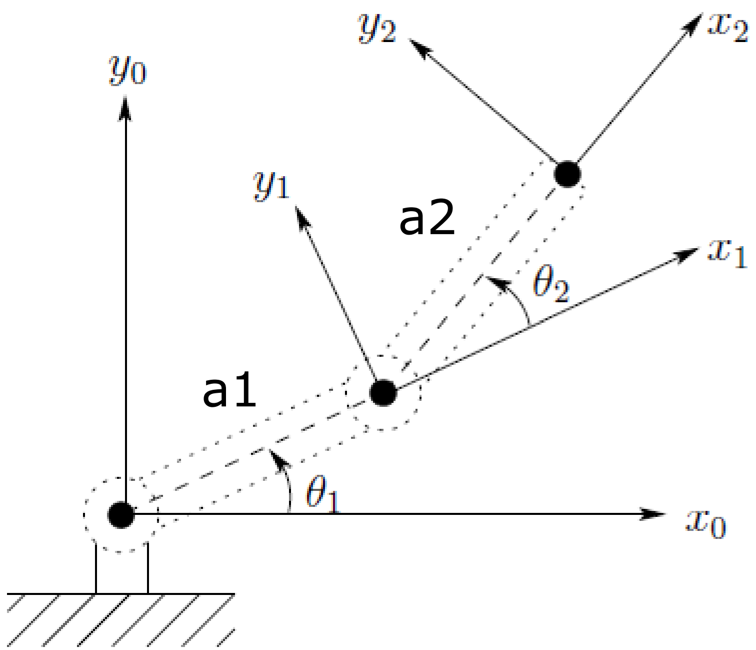
$$\dot{x} = \begin{bmatrix} -\alpha_1 \sin\theta_1 - \alpha_2 \sin(\theta_1 + \theta_2) & -\alpha_2 \sin(\theta_1 + \theta_2) \\ \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1 + \theta_2) & \alpha_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \dot{\theta}$$

Two-Link Planar Robot Arm

- Using the vector notation

$$\mathbf{v} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \text{ and } \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= -\alpha_1 \sin\theta_1 \cdot \dot{\theta}_1 - \alpha_2 \sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ \dot{y} &= \alpha_1 \cos\theta_1 \cdot \dot{\theta}_1 + \alpha_2 \cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \end{aligned}$$



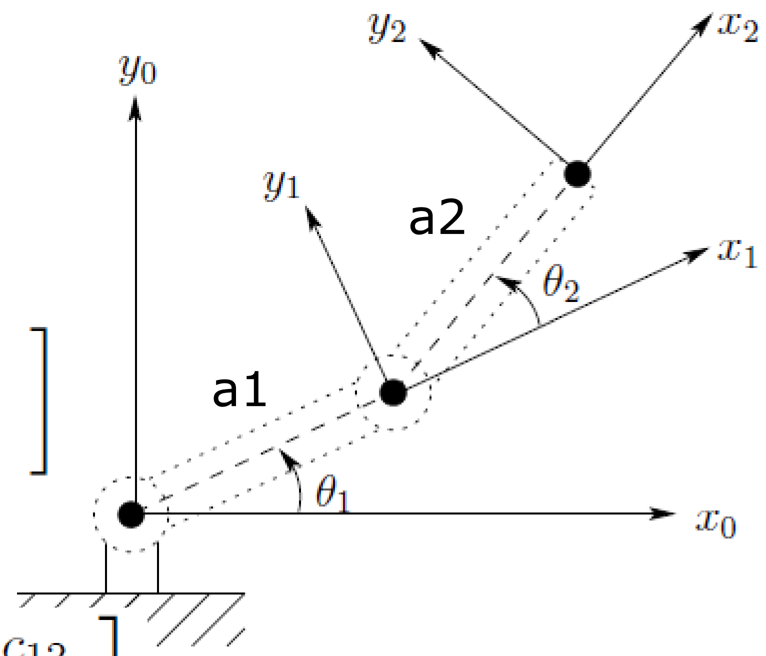
$$\mathbf{v} = \begin{bmatrix} -\alpha_1 \sin\theta_1 - \alpha_2 \sin(\theta_1+\theta_2) & -\alpha_2 \sin(\theta_1+\theta_2) \\ \alpha_1 \cos\theta_1 + \alpha_2 \cos(\theta_1+\theta_2) & \alpha_2 \cos(\theta_1+\theta_2) \end{bmatrix} \dot{\theta} = J_v \dot{\theta}$$

- The angular speed of tool is

$$\mathbf{w} = \dot{\theta}_1 + \dot{\theta}_2 = J_\omega \, \dot{\theta}$$

Derivation of Jacobian for Two-Link Planar Robot Arm

- Since there are two joints size of Jacobian matrix must be 6 × 2
- Since the joints are revolute, form of the Jacobian must be



$$J(q) = \begin{bmatrix} z_0 \times (o_2 - o_0) & z_1 \times (o_2 - o_1) \\ z_0 & z_1 \end{bmatrix}$$

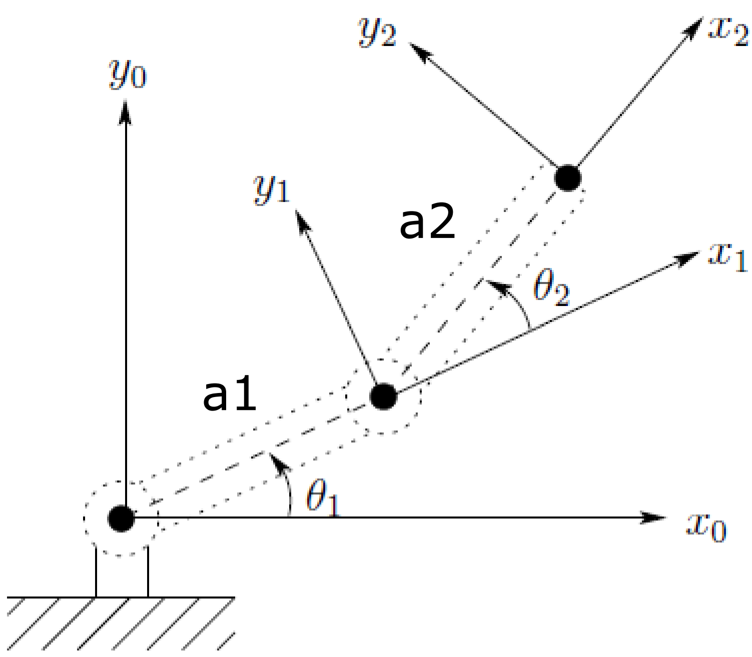
$$o_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$z_0 = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Derivation of Jacobian for Two-Link Planar Robot Arm

- Performing the required calculations then yields

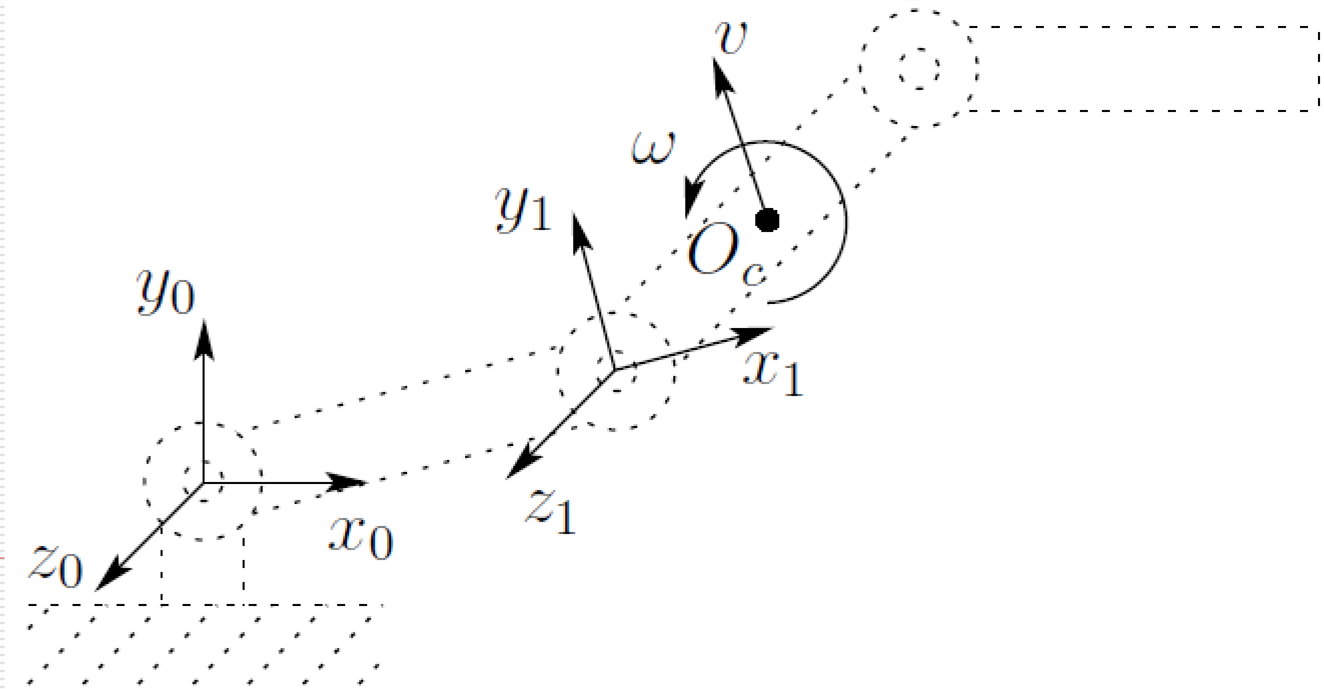
$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



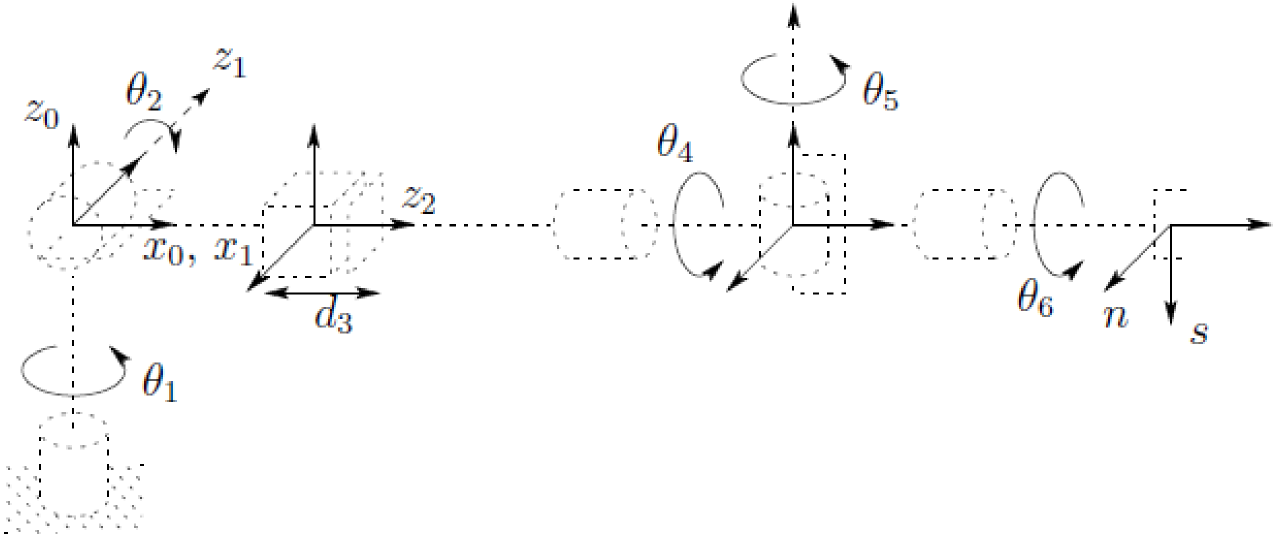
Example 4.6
Jacobian for an Arbitrary Point

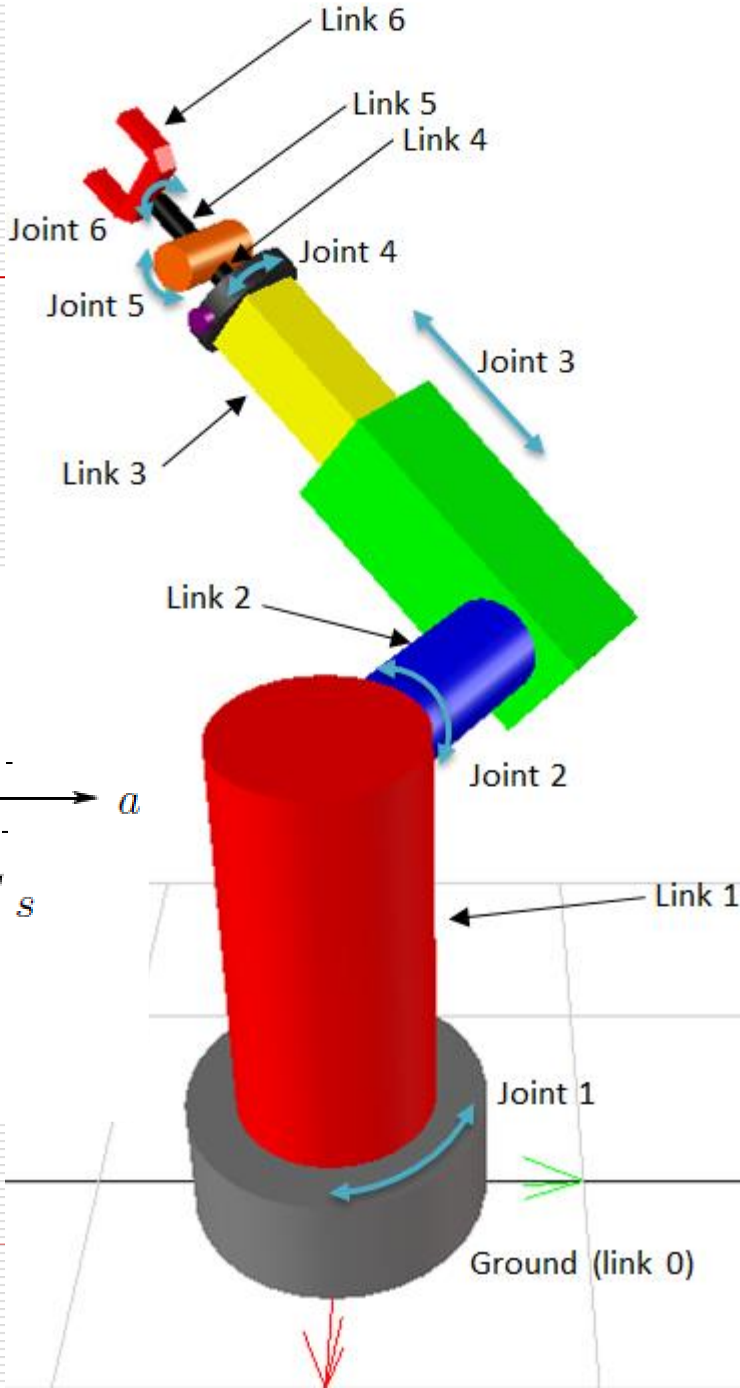
- Consider the three-link planar manipulator.
- To compute the linear velocity v and the angular velocity w of the center of link 2 as shown.

$$J(q) = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix}$$



Stanford Manipulator With A Spherical Wrist

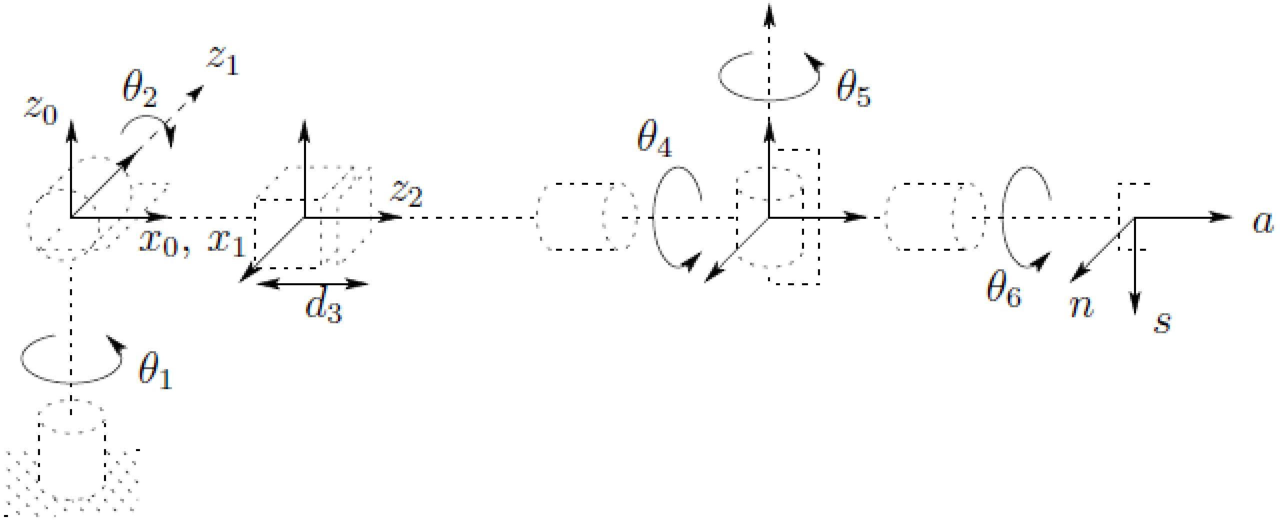




DH parameters for Stanford Manipulator

Link	d_i	a_i	α_i	θ_i
1	0	0	-90	$\theta^\star$
2	d_2	0	$+90$	$\theta^\star$
3	$d^\star$	0	0	0
4	0	0	-90	$\theta^\star$
5	0	0	$+90$	$\theta^\star$
6	d_6	0	0	$\theta^\star$

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

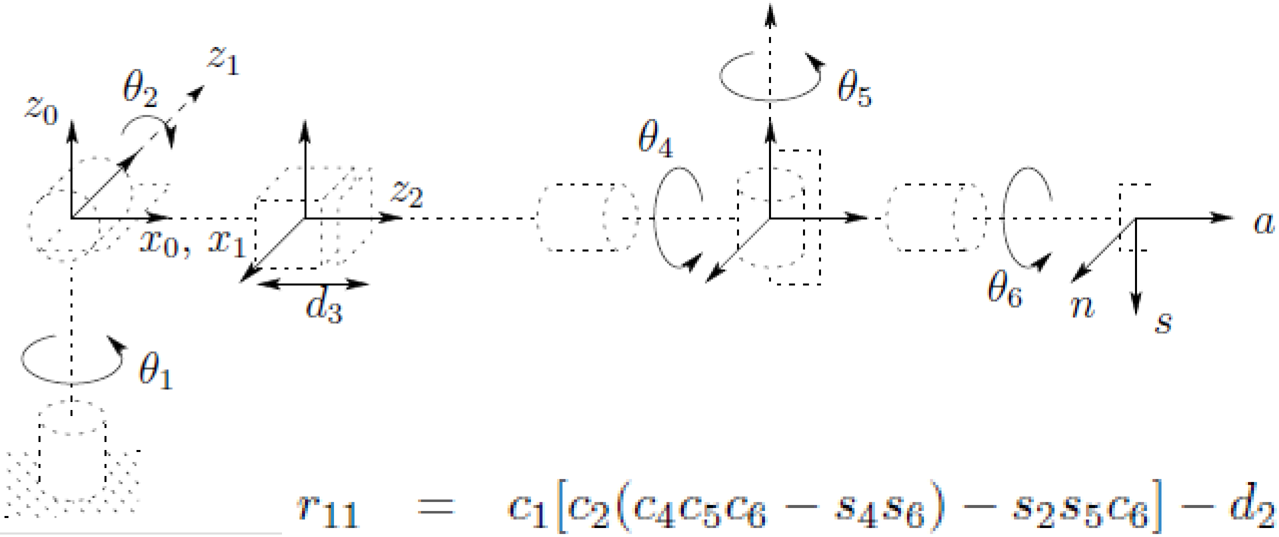
$$A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}$$

Derivation of Jacobian

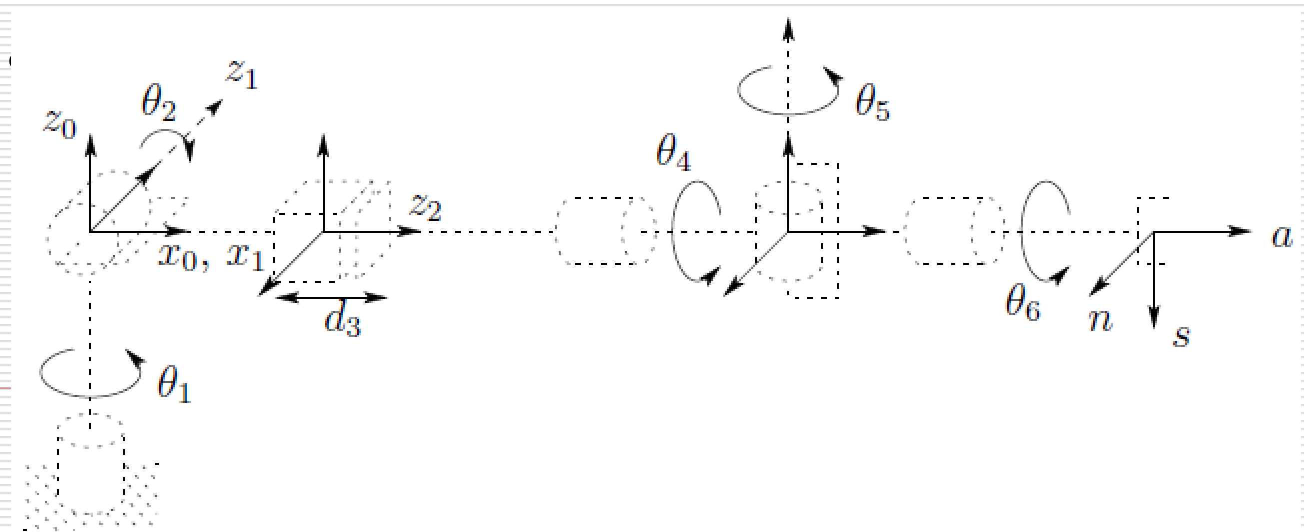
- Note that o1=o2, joint 3 is prismatic and that o3 = o4 =o5 as a consequence of the spherical wrist and the frame assignment.
- First, oj is given by the first three entries of the last column of T_j^0= A1 · · ·Aj , with o0 = (0, 0, 0)^T = o1.
- The vector z_j is given

$$z_j = R_j^0 k$$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o_{i-1}) \\ z_{i-1} \end{bmatrix} \quad i = 1, 2$$

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o) \\ z_{i-1} \end{bmatrix} \quad i = 4, 5, 6$$



$$o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 - c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

$$o_3 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

The z_i are given as

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$$

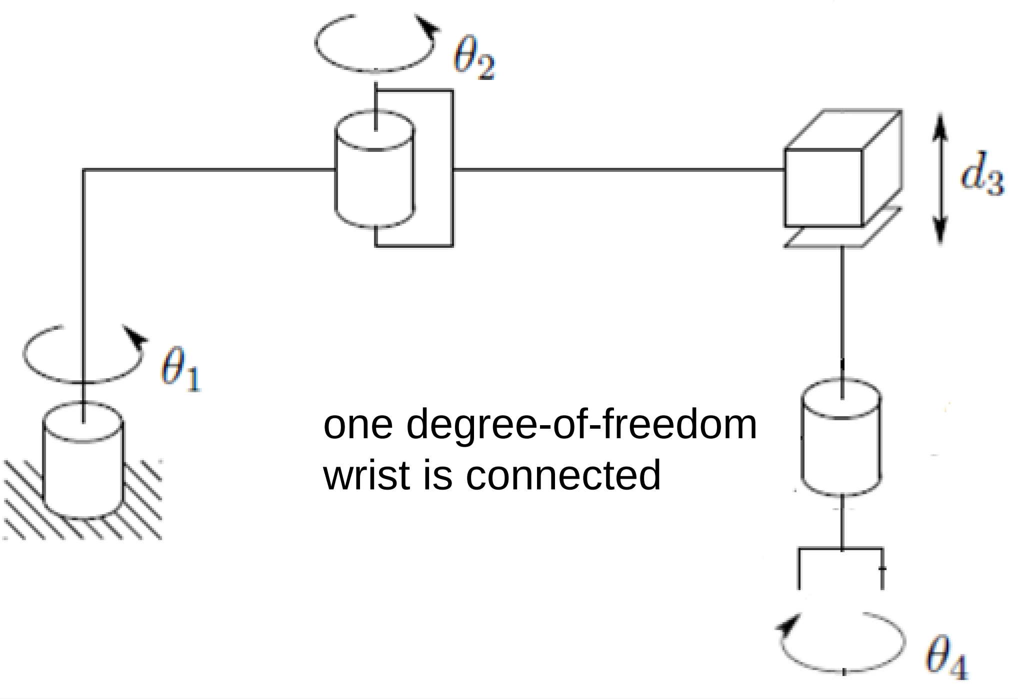
$$z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \qquad z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix}$$

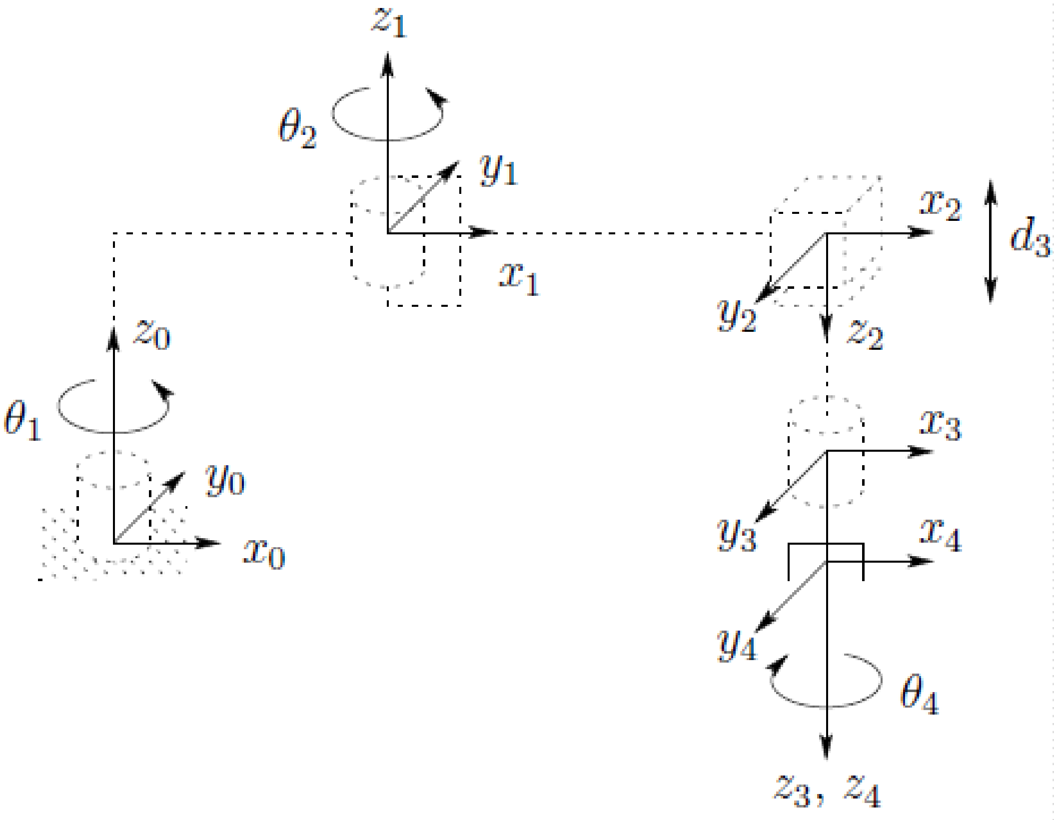
$$z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}.$$

SCARA Manipulator

The Epson E2L653S SCARA Robot

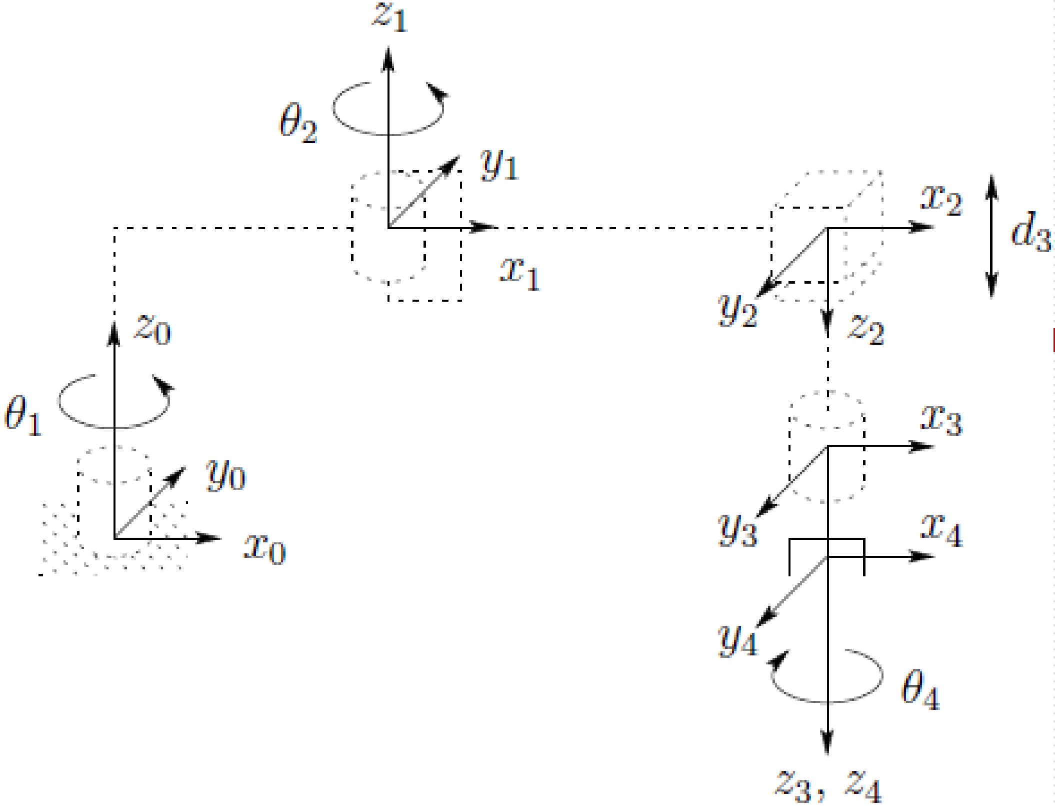


The SCARA (Selective Compliant Articulated Robot for Assembly).



DH parameters for SCARA

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$



Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2 c_2 \\ s_2 & -c_2 & 0 & a_2 s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

forward kinematic equations

$$\begin{aligned} T_4^0 &= A_1 \cdots A_4 \\ &= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	$\theta^\star$
2	a_2	180	0	$\theta^\star$
3	0	0	$d^\star$	0
4	0	0	d_4	$\theta^\star$

Derivation of Jacobian For SCARA Manipulator

- This Jacobian is a 6 × 4 matrix since the SCARA has only four degrees-of freedom.
- $o_0=[0\ 0\ 0]^T$, $o_1=?$, $o_2=?$, $o_4=?$
- o_1 is the first three elements of last column of A_1 matrix

$$o_1 = \begin{bmatrix} a_1c_1 \\ a_1s_1 \\ 0 \end{bmatrix}$$

- o_2 is the first three elements of last column of T_2^0=A_1 A_2

$$o_2 = \begin{bmatrix} a_1c_1 + a_2c_{12} \\ a_1s_1 + a_2s_{12} \\ 0 \end{bmatrix}$$

- o_4 is the first three elements of last column of T_4^0

$$\begin{aligned} T_4^0 &= A_1 \cdots A_4 \\ &= \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly $z_0 = z_1 = k$, *and* $z_2 = z_3 = -k$. *Therefore the Jacobian of the SCARA Manipulator is*

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$